Inventory Management for Customers with Alternative Lead-time Choices

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Abstract

This paper considers an inventory model where one supplier provides alternative lead-time choices for customers: the short and the long lead-time. We obtain the optimal dynamic inventory-commitment policy for the supplier, that is when to deliver long lead-time requests. We also prove that the optimal inventory replenishment policy is a base-stock type. The optimal commitment levels are independent from the purchasing cost, the salvage value, the long and short lead-time prices; and robust to the inventory holding cost. Further, we compare the profit between the optimal dynamic inventory-commitment policy with the static inventory rationing policy. In addition, we use the customer choice model to characterize the risk-pooling, demand-induction and -cannibalization effects; with both analytical results and numerical experiments, we demonstrate the profit improvement by allowing a dynamic inventory-commitment policy.

1 Introduction

A supplier/manufacturer operating in a batch production mode usually deals with two kinds of customers. Customers with an advanced forecasting capability and relatively stable demand may operate in a make-to-stock mode. Their planning process is with a cyclic patten, and the cycle can be slower (longer) than the supplier. Customers with a conventional forecasting capability and unstable demand may operate in a make-to-order mode. Their planning process is triggered by sales orders. Therefore, their planning cycle can be faster (shorter) than the supplier. To the supplier, orders from slow customers can be promised and included in the next production cycle. However, orders from fast customers have to be satisfied from on-hand inventory.

A batch mode of operation is common in both manufacturing and distribution sectors. Semiconductor wafer fabrication (Gurnani, Anupindi and Akella (1992)) is a multi-stage process including operations such as photolithography, diffusion, metallization. Owning to the complexity of the wafer fabrication process, existence of batch operational machines, pilot runs and uncertain yield rate, wafer fabrication is carried out in a batch mode. Immediate orders for a specific integrate circuit can only be satisfied from its on-hand inventory while future delivery orders can be included into the next production cycle.

Distribution operation can also be in a batch mode. Tropicana Juice is a division of PepsiCo, Inc. In the morning of each working day, Tropicana decides how many units of juice products to be carried on its specially-modified refrigerated trucks to its distribution centers in New Jersey. Wholesalers dispatch trucks to the center to replenish their inventories. Tropicana promises to satisfy early booked orders and rush-orders on the available bases.

An acclaimed management concept in service industry is the notion of revenue management, where capacities are reserved for possible high margin custoemrs while the effort is trying to fill all capacities available. Similarly, in the manufacturing sector, order promissing, or ATP (available to promise), is the communication process between a supplier and a customer for a reliable delivery date. Modern manufacturing companies rely on their ERP (Enterprise Resource Planning) systems or APS (Advanced Planning Systems) to provide a real-time order delivery information. I2 Technologies claims that: "recognizing ATP as material and capacity that can be combined to build new products (a technique called capable-to-promise) is a powerful order promising feature that provides forward-looking visibility of what can be produced and promised, as well as maximum production flexibility by delaying the commitment of a particular end item as long as possible."

In this paper, we define a *long lead-time customer* as the one who accepts a set of two or more alternative lead-times so that the supplier can choose the delivery time at the supplier's convenience. On the other hand, a *short lead-time customer*'s demand needs to be met immediately. The supplier has its own production lead-time, which is shorter than the lead-time of long lead-time customers. In this case, the supplier decides when to deliver the promised orders to maximize its profits. Once an order is placed, short lead-time customers are informed if the orders is accepted, that is, to deliver products now, while long lead-time customers are infomed when the orders will be delivered. And long lead-time customers are guaranteed for a service since the supplier has a sufficient capacity to replenish its inventory.

1.1 Literature review

Gallego and Phillips (2004) introduce the concept of flexible products into a two-stage airline revenue management problem. The flexible product is defined as a set of two alternative flights serving the same market, and the flexible product can only be exercised in the first stage. They derive the conditions and algorithms for the management of a single flexible product consisting of two specific products. Gallego et al. (2006) extend the problem into multi-period problem without any restrictions on the types and the times of the request arrivals. For an inventory model with a flexible manufacturing machine, Chen (2004) considers a flexible production system where a flexible machine produces two products. He proves the optimality of the hedging point policies in a periodic review system.

Another related literature is about an inventory system which serves multiple customer classes. The representative strategy is known as *inventory-rationing policies*. Inventory-rationing policy chooses a static value for each customer class, which defines the maximum number of orders would be satisfied for each customer class. Veinott (1965) firstly considers the problem of multiple demand classes and introduces the concept of rationing. Topkis (1968) develops a dynamic inventory-rationing policy where he divides the period into a finite number of intervals. On-hand inventory is allocated to each time-interval. In each time-interval, orders are accepted or backlogged (lost) at the end of interval, where an available-to-promise (ATP) decision is done in a batch format. Kaplan (1969) obtains the same inventory-rationing policy for two demand classes, without considering the inventory-replenishment issue. Ha studies a manufacturing system operating a make-to-stock environment with an inventory-rationing policy (1997a and 1997b). Recently, Cattani and Souza (2002) study continous-reveiw inventory management with customers of two lead-time requirements in comparing the performance of the static inventory-rationing policy and the first-come first-serve policy by numerical results.

There is a large body of literature about both deterministic and stochastic lead-time issues. Fukuda (1964) firstly studies the multi-period inventory problem with two deterministic lead-time delivery. Kaplan (1970) studies a dynamic inventory problem with random delivery lead-times. Ehrhardt (1984) works on an infinite horizon model with stochastic lead-times and obtains the optimal policy for minimizing the discounted and the average cost. Song and Zipkin (1996) extend the model to an evolving system. Moreover, the effect of lead-time uncertain is examined when the performance measure of interest is the long-run average cost (Song (1994a)), as well as the total discounted cost (Song (1994b)). Sethi, Yan and Zhang (2003, and 2005), and Feng et al. (2006) study a class of models of inventory decision with multiple delivery lead-time and information updates. This line of research characterizes the conditions for familiar base-stock, and (s, S) policies remain to be optimal.

1.2 Main results and plan of the paper

In this paper, motivated by our consulting work with Suga, we introduce the concept of flexible lead-time into an inventory management model, incorporating an inventory-replenishment decision at each planning cycle. The notion of lead-time choice requires a rigorous analysis should suppliers offer the lead-time flexibility.

Researchers in revenue management have been studying what to deliver and how to price should a flexible product exists, such as the work by Gallego and Phillips (2004). Moreover, owing to the nature of perishable products and fixed capacity, models in revenue management are single period model in general, which do not involve with inventory dynamics nor capacity optimization. Therefore, dealing with both when to commit on-hand inventory and how many units of inventory to hold are essential for non-perishable products with lead-time choices.

This paper develops a multi-period inventory model, where one supplier provides two lead-time choices to its customers. We first characterize the optimal inventory-commitment policy, i.e. to decide when to use the on-hand inventory to satisfy a long lead-time customer. We further prove that the inventory replenishment policy at the beginning of each cycle is a base-stock policy. The optimal inventory-commitment policy and the optimal inventory replenishment policy provide us a foundation to further characterize the lead-time choice inventory system. We compare the optimal dynamic inventory commitment policy with the inventory rationing policy. We investigate the risk-pooling, demand-induction and -cannibalization effects.

In the next section, we describe the problem and provide notations of the paper. In Section 3, we obtain an optimal inventory commitment policy. In Section 4, we first prove that the optimal inventory replenishment policy is a base-stock one. Then we extend the above results to the multi-period case. We compare the optimal dynamic inventory-commitment policy with an inventory rationing policy in Section 5. We demonstrate the risk pooling, and demand induction and cannibalization effects in Section 6 and 7, respectively. Finally, we conclude the paper and briefly discuss the furture research directions in Section 8.

2 Problem Description and Notations

We consider an inventory management problem with one supplier who provides two delivery options to its customers: delivery now or in the next cycle. When customers arrive, based on customers' preference, the supplier commits its inventories in responding to its customers. When the customer requires a short lead-time delivery, the supplier has to satisfy the request with on-hand inventories if the supplier has inventory on hand. When the customer requires a long lead-time delivery, the supplier has an option to satisfy the customer now or delay the order to the next cycle. The sequence of events is illustrated in Figure 1.

At the beginning of each cycle, the supplier replenishes its inventory, and sets the same product with two prices with respect to the delivery lead-time. Similar to techniques used by Gallego et al. (2006), and Talluri and Ryzin (2004), we divide a cycle into small sub-periods $\{1, 2, \cdots, T\}$. It is assumed a sub-period is small enough such that at most one customer arrives in each sub-period. Specifically, in each sub-period, the short lead-time customer, with a probability of π_s, places an order and expects the order to be delivered immediately. Demands from these short lead-time customers are lost if their orders cannot be satisfied immediately. With a probability

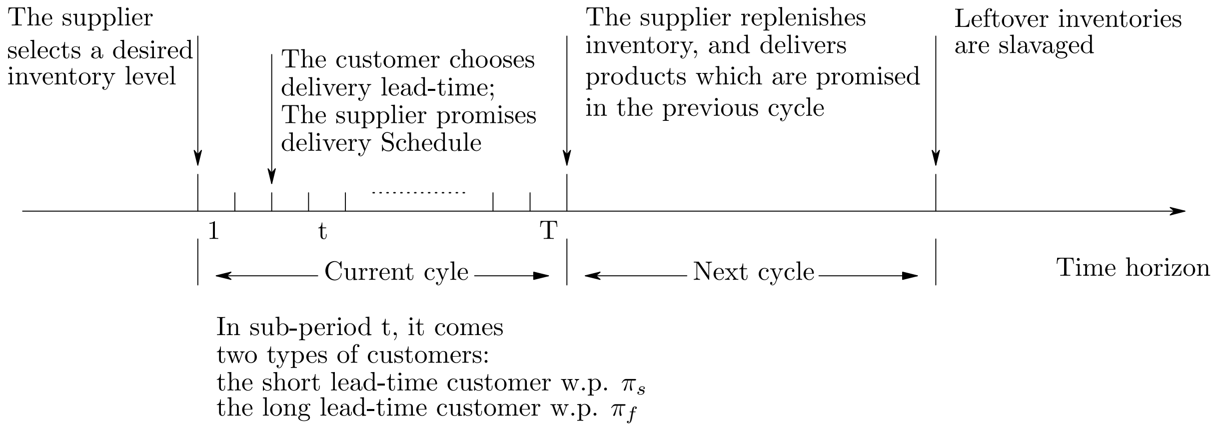

Figure 1: The Sequence of the Events and Decisions

π_f, the long lead-time customer places an order and expects the order to be delivered no later than the next cycle. The short and long lead-time customers pay a unit price p_s and p_f ($p_f \leq p_s$), respectively. Note that we assume that the supplier has a sufficient capacity, such that all long lead-time customers can be satisfied in the next cycle. Denote also π_0 as the probability of no orders in each sub-period. Obviously, we have $\pi_s + \pi_f + \pi_0 = 1$. At the end of each cycle, leftover inventories are carried over to the next cycle, and a unit inventory holding cost h applies to each unit of inventories. Leftover inventories can be salvaged at the unit price of s.

At the beginning of the next cycle, the supplier replenishes its inventory and fulfills all committed orders for long lead-time customers. In the last cycle, leftover inventories are salvaged at the unit price of s.

To faciliate the analysis, we denote the on-hand inventory and the amount of products promised to deliver in the next period as $n(t)$ and $m(t)$, respectively, where t represents the tth sub-period of the cycle. The inventory-commitment policy for long lead-time customers at sub-period t is

$$u(t) = \begin{cases} 1, & \text{the long lead-time order to be satisfied with on-hand inventory} \\ & \text{and delivered immediately;} \\ 0, & \text{the long lead-time order to be promised to be delivered in the next cycle.} \end{cases}$$

To maximize the profit in this inventory model with alternative lead-time choices, it is necessary for the supplier to find an optimal dynamic inventory-commitment policy $u(t)$, with respect to $n(t)$, $m(t)$ and the elapsed sub-period t. With an optimal inventory commitment policy, it is possible for us to address the issue of the optimal inventory stocking, i.e. to characterize the optimal inventory replenishment policy. The former is solved in Section 3, and the latter is discussed in Section 4.

3 The Optimal Inventory Commitment Policy

Since the price for the short lead-time customer p_s is no less than the price for the long lead-time customer p_f, it is always beneficial for the supplier to meet the short lead-time order as long as the on-hand inventory is positive. On the other hand, it is critical to decide when to fulfill orders from the long lead-time customers. In this section, we present a model to derive the optimal inventory-commitment policy.

Let $V(t, n, m)$ represent the value function from sub-period t when the amounts of the on-hand inventory and the promised-to-delivery product are n and m, respectively. When $n(t) = 0$, neither long lead-time nor short lead-time customers can be satisfied in the current period. Therefore, the demand for the short lead-time customers is lost, and orders for long lead-time customers are scheduled to be delivered in the next cycle. The supplier obtains a profit of p_f from a long lead-time customer, that is for $1 \le t < T$,

$$V(t, 0, m) = (\pi_0 + \pi_s)V(t + 1, 0, m) + \pi_f(p_f + V(t + 1, 0, m + 1)), \tag{1}$$

and

$$V(T, 0, m) = -cm, \tag{2}$$

Where c is the unit purchasing cost. With some algebraic calculations, it is possible to derive that the value function for zero on-hand inventory is

$$V(t, 0, m) = (T - t)\pi_f(p_f - c) - cm. \tag{3}$$

For $n(t) \ge 1$, the supplier decides when the long lead-time customer can be satisfied. The dynamic programming equation is, for $1 \le t < T$,

$$\begin{aligned} V(t, n, m) = {} & \pi_0 V(t + 1, n, m) + \pi_f \max_{u \in \{0,1\}} \{V(t + 1, n - u, m + 1 - u)\} \\ & + \pi_s V(t + 1, n - 1, m) + \pi_f p_f + \pi_s p_s; \end{aligned} \tag{4}$$

and

$$V(T,n,m) \quad = \quad -c(m-n)^+ - hn + s(n-m)^+. \tag{5}$$

where the unit salvage value s is assumed to be less than the unit purchasing cost c.

Remark 3.1 *Although our results are derived in the discrete time setting, they can be easily extended to a continuous time model. In this case, the value function is*

$$\begin{aligned} V(t,n,m) &= \max_u E\{\int_t^T [\pi_s p_s 1_{n(s^-)>0} + \pi_f p_f 1_{n(s^-)>0} u(s) + \pi_f p_f (1-u(s))] dN(s) \\ &\quad -hn(T) - c(m(T)-n(T))^+\} \\ &= \max_u \lambda\{\int_t^T [\pi_s p_s 1_{n(s^-)>0} + \pi_f p_f 1_{n(s^-)>0} u(s) + \pi_f p_f (1-u(s))] ds \\ &\quad -hEn(T) - cE(m(T)-n(T))^+\}, \end{aligned} \tag{6}$$

where $N(t)$ stands for the accumulated amount of customer arrivals until time t, which is a homogeneous Poisson process with a constant intensity λ, and s^- denotes the time just before s. Let

$$\begin{aligned} F(\Delta t) &= \max_u\{\lambda\Delta t(\pi_s p_s 1_{n(t)>0} + \pi_f p_f 1_{n(t)>0} u(t) + \pi_f p_f (1-u(t))) \\ &\quad + \int_{t+\Delta t}^T [\pi_s p_s 1_{n(s^-)>0} + \pi_f p_f 1_{n(s^-)>0} u(s) + \pi_f p_f (1-u(s))] ds \\ &\quad -hEn(T) - cE(m(T)-n(T))^+\} \\ &= \max_u\{\lambda\Delta t(\pi_s p_s 1_{n(t)>0} + \pi_f p_f 1_{n(t)>0} u(t) + \pi_f p_f (1-u(t))) \\ &\quad +V(t+\Delta t, n(t+\Delta t), m(t+\Delta t))\}, \end{aligned} \tag{7}$$

which converges to $V(n,m,t)$, when $\Delta t \to 0$. The state transitions are $n(t+\Delta t) = n(t) - \lambda\Delta t(\pi_s + \pi_f u(t))$ and $m(t+\Delta t) = m(t) + \lambda\Delta t\pi_f(1-u(t))$. Hence, we can obtain that, for $n(t) > 0$,

$$\begin{aligned} F(\Delta t) &= \lambda\Delta t(\pi_s p_s + \pi_f p_f) + \max_u V(t+\Delta t, n(t+\Delta t), m(t+\Delta t)) \\ &= \lambda\Delta t(\pi_s p_s + \pi_f p_f) + \max\{V(n(t) - \lambda\Delta t(\pi_s + \pi_f), m(t), t+\Delta t), \\ &\quad V(n(t) - \lambda\Delta t\pi_s, m(t) + \lambda\Delta t\pi_f, t+\Delta t)\}, \end{aligned} \tag{8}$$

which possesses a similar structure as Equation (4). Then by the same procedure of what follows in this paper, it is easy to prove that $F(\Delta t)$ preserves all of our results, for any Δt. Let $\Delta t \to 0$, and

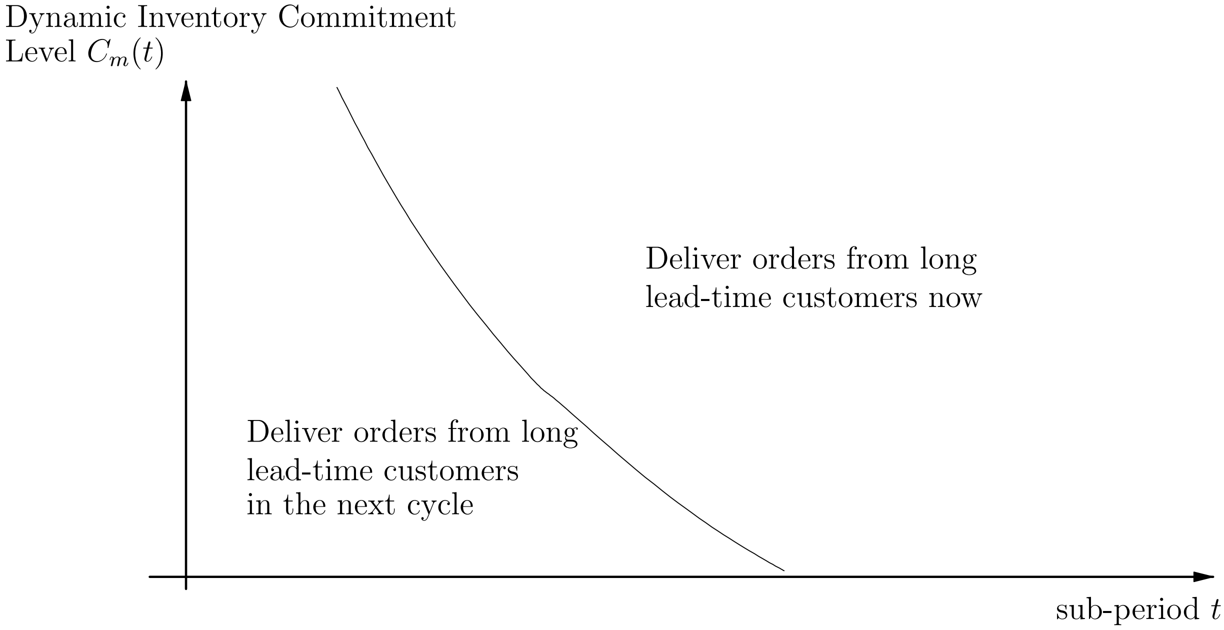

Figure 2: An Illustration of the Optimal Inventory Commitment Policy

we obtain that our results, which we obtain in what follows, can be extended into the continuous time setting.

Equation (4) indicates that it is necessary to compare the profit changes from two possible actions: to deliver one unit of product to the long lead-time customer now, or to delay it to the next cycle. And the following lemma provides a monotonicity property of the profit difference between these two actions. Its proof appears in Appendix.

Lemma 3.1 (*a*)*For a given m, $V(t,n-1,m)-V(t,n,m+1)$ is non-decreasing in n; i.e. $V(t,n-1,m) - V(t,n,m+1) \leq V(t,n,m) - V(t,n+1,m+1)$.*

(*b*) *For a given n, $V(t,n-1,m) - V(t,n,m+1)$ is independent from m; i.e. $V(t,n-1,m) - V(t,n,m+1) = V(t,n-1,m+1) - V(t,n,m+2)$.*

We now present the following theorem in characterizing the optimal dynamic inventory-commitment policy, which is a function of the on-hand inventory level $n(t)$ as well as the sub-period t. Its proof appears in Appendix.

Theorem 3.1 (1) *The optimal inventory-commitment policy is characterized by a switching manifold. The manifold $C_m(t)$ equals to 0 if $V(t+1,0,m) - V(t+1,1,m+1) \geq 0$; and is defined* *as the maximum value of n such that $V(t+1, n-1, m) - V(t+1, n, m+1) \leq 0$, otherwise. Moreover, the switching manifold is independent of the next cycle promised order quantity m, and non-increasing with respect to the time t.*

(2) *In each sub-period, the optimal control is: to deliver orders from long lead-time customers now if $n(t) > C_m(t)$; otherwise to promise long lead-time customers that their orders will be delivered in the next cycle. The inventory-commitment policy is illustrated in Figure 2.*

This theorem provides the optimal inventory-commitment policy when the supplier faces order from a long lead-time customer. The supplier balances the trade-off between delivering products now and delaying the orders to the next cycle: the former may save on-hand inventory to meet potential short lead-time orders in the remaining of the cycle; the latter may reduce the on-hand inventory to get rid of a potential inventory holding cost. The result of the trade-off is to follow a dynamic inventory-commitment policy in Theorem 3.1.

4 The Optimal Inventory Replenishment Policy

In the last section, we have obtained the optimal inventory-commitment policy for the supplier during the selling cycle. In this section, we extend our study to investigate the existence and the form of the optimal inventory replenishment policy. We start from a single cycle problem.

4.1 Optimal Inventory Replenishment Policy for a Single Cycle Problem

To facilitate the discussion in this section, we define $\hat{V}$ as the set of functions on $f : N \times N \longrightarrow R$ where N stands for $\{0, 1, 2, \cdots\}$, and if $V \in \hat{V}$, then:

P1 $V(t, n, m)$ is concave in n, for a given m;

P2 $V(t, n, m)$ is concave in m, for a given n;

P3 $V(t, n, m) - V(t, n-1, m)$ is non-decreasing in m, for a given n; and $V(t, n, m+1) - V(t, n, m)$ is non-decreasing in n, for a given m.

The following lemma claims that $V(t, n, m) \in \hat{V}$. Its proof appears in Appendix.

Lemma 4.1 *The value function $V(t,n,m) \in \hat{V}$, i.e. $V(t,n,m)$ satisfies $P1$, $P2$ and $P3$.*

At the beginning of the selling season, the supplier has an intial inventory $n(0)$. A desired on-hand inventory level is selected as $n(1)$ s.t. $n(1) \geq n(0)$ to maximize profits, i.e.

$$\max_{n(1)\geq n(0)} (V(1,n(1),0) - c(n(1)-n(0))) = -cn(0) + \max_{n(1)\geq n(0)} (V(1,n(1),0) - cn(1)),$$

where $c(n(1)-n(0))$ is the purchasing cost. This can be easily solved since $V(1,n(1),0)$ satisfies the property of P1. The result can be characterized as a base-stock policy as follows.

Theorem 4.1 *The inventory replenishment policy is a base-stock type, i.e. there exists an optimal order-up-to level y^* such that the optimal ordering quantity is*

$$q^* = \begin{cases} y^* - n(0), & \text{if } n(0) \leq y^*, \\ 0, & \text{if } n(0) > y^*; \end{cases}$$

where y^* is the maximizer of $V(1,n(1),0) - c(n(1)-n(0))$, which is independent from $n(0)$.

Then, with a simple search algorithm, it is possible for us to find the optimal stocking level.

4.2 Extensions to a Multi-cycle Problem

In this subsection, we extend our results into a multi-cycle problem. In the model we discussed before, after meeting the products at the beginning of the next cycle, the left products are salvaged. In the multi-cycle problem, the leftover poducts can be considered as the initial inventory of the consequentive cycle. Then at the beginning of each cycle (except the first cycle), the inventory replenishment have two functions: (1) to satisfy the orders promised in the last cycle, and (2) to setup an initial inventory for orders in the future. Without a capacity constraint, these two parts can be separated, because the orders promised in the last cycle are known when the replenishment decision is made. Let i be the cycle index. For the last cycle $i = I$, the value function $V_I(T,n,m)$ is the same as Equation (5). For cycle $i < I$, the leftover inventories are carried over to the next cycle, and the value function can be rewritten as

$$\begin{aligned} V_i(T,n,m) &= -c(m-n)^+ - hn + \max_{y\geq(n-m)^+} \{-c(y-(n-m)^+) + V_{i+1}(1,y,0)\} \\ &= c(n-m) - hn + \max_{y\geq(n-m)^+} \{-cy + V_{i+1}(1,y,0)\}, \qquad (9) \end{aligned}$$

where y is the inventory position after replenishment at the beginning of cycle $i+1$. Other dynamic equations can be carried out similarly. Now we present the following theorem.

Theorem 4.2 *For a multi-cycle problem, the delivery policy for long lead-time customers in each cycle is the dynamic inventory-commitment policy, characterized by Theorem 3.1.*

Given the leftover inventory n and promised-to-delivery product m from the previous cycle, the inventory replenishment at the beginning of cycle i consists of the following two parts: (1) order $(m-n)^+$ to satisfy the promised orders from the previous cycle; (2) order up to a base-stock level y^*, i.e. order $(y^* - (n-m)^+)^+$, where y^* is the maximizer of $V_i(1, y, 0) - c(y - (n-m)^+)$.

Proof. Recall that the properties of (a) and (b) in Lemma 3.1, and P1, P2 and P3 in Lemma 4.1 are required to construct the optimal policy. It is possible for us to demonstrate that $V_I(T, n, m)$ satisfies (a) and (b), P1, P2 and P3, and the derivative of $V_I(T, n, 0)$ with respect to n is no greater than c. Now suppose that $V_{i+1}(T, n, m)$ satisfies the above mentioned conditions. Then, we varify if $V_i(T, n, m)$ preserves these properties.

From the proof of Lemma 3.1 and 4.1, we obtain that $V_{i+1}(1, n, m)$ satisfies (a), (b) and P1, P2, P3. Since we have assumed that the derivative of $V_{i+1}(T, n, 0)$ in n is no greater than c, by Equation (4), it is straight forward to obtain that the derivative of $V_{i+1}(1, n, 0)$ in n is no greater than c, i.e.

$$V_{i+1}(1, n, 0) - V_{i+1}(1, n-1, 0) \quad \leq \quad c \tag{10}$$

By the concavity of $V_{i+1}(1, y, 0)$ in y (property P1), we obtain that there exists a maximizer y^* of $-cy + V_{i+1}(1, y, 0)$, and (9) can be rewriten as

$$V_i(T, n, m) \quad = \quad c(n-m) - hn - c(y^* \vee (n-m)^+) + V_{i+1}(1, y^* \vee (n-m)^+, 0), \tag{11}$$

where $y^* \vee (n-m)^+$ equals to the maximum of y^* and $(n-m)^+$. From (10) and (11), it is staight forward that the derivative of $V_i(T, n, 0)$ in n is no greater than c.

From (11), we obtain that $V_i(T, n-1, m) - V_i(T, n, m+1) = h$, that is, $V_i(T, n, m)$ satisfies (a) and (b). In what follows, we prove that $V_i(T, n, m)$ satisfies P1, P2, and P3 step by step. To facilitate the proof, we denote

$$y^* \vee (n-m)^+ \quad = \quad \begin{cases} n-m, & \text{if } n \geq m + y^*; \\ y^*, & \text{otherwise.} \end{cases} \tag{12}$$

First let us fix m. When $n \geq m + y^*$, then $V_i(T, n, m) = c(n-m) - hn - c(n-m) + V_{i+1}(1, n-m, 0)$ which is concave in n, since $V_{i+1}(1, n, m)$ satisfies P1. The derivative of $V_i(T, n, m)$

with respect to n is no greater than $-h + c$ by (10). When $n \leq m + y^* - 1$, $V_i(T, n, m) = c(n - m) - hn - cy^* + V_{i+1}(1, y^*, 0)$ which is concave in n, and the derivative is $-h + c$. Hence $V_i(T, n, m)$ is concave in n, i.e. it satisfies P1.

Then let us fix n. By (10), we obtain that the derivative of $V_{i+1}(1, n - m, 0)$ in m is no less than $-c$. When $m \leq n - y^*$, then $V_i(T, n, m) = c(n - m) - hn - c(n - m) + V_{i+1}(1, n - m, 0)$ which is concave in m, since $V_{i+1}(1, n, m)$ satisfies P2. Its derivative with respect to m is no less than $-c$. When $m \geq n - y^* + 1$, $V_i(T, n, m) = c(n - m) - hn$ which is concave in m, and its derivative is $-c$. Hence $V_i(T, n, m)$ is concave in m, i.e. it satisfies P2.

When $m \leq n - y^*$, then $V_i(T, n, m) - V_i(T, n - 1, m) = -h + V_{i+1}(1, n - m, 0) - V_{i+1}(1, n - 1 - m, 0)$. $V_{i+1}(1, n - m, 0) - V_{i+1}(1, n - 1 - m, 0)$ is non-decreasing in m, since $V_{i+1}(T, n, m)$ satisfies P1. Hence, $V_i(T, n, m) - V_i(T, n - 1, m)$ is non-decreasing in m. And it is no greater than $-h + c$ by (10). When $m \geq n - y^* + 1$, $V_i(T, n, m) - V_i(T, n - 1, m) = -h + c$. Hence $V_i(T, n, m) - V_i(T, n - 1, m)$ is non-decreasing in m, i.e. $V_i(T, n, m)$ satisfies P3. To this end, we have proved that $V_i(T, n, m)$ satisfies P1, P2 and P3. □

5 Study of the Static Inventory Rationing and the Dynamic Inventory-Commitment Policy

In Theorem 3.1, we prove that the optimal inventory commitment policy depends on a time-dependent switching manifold for long lead-time customers. In the literature, inventory rationing policy assumes a constant level for different demand classes. Cattani and Souza (2002) compare the performance of the inventory rationing policy with a first-come first-serve policy. They conclude that the inventory rationing policy out performs the first-come first-serve policy. In this section, we carry out a study in comparing the performance of the dynamic inventory-commitment policy to the static inventory rationing policy.

Note that for the dynamic inventory-commitment policy, according to Theorem 3.1, consists of one optimal commitment level $C_m(t)$ in each sub-period t, and the commitment levels vary in different sub-periods (see Figure 3 for its illustration). The supplier delivers the long lead-time customer a product in the current cycle as long as the inventory level is greater than the commitment level at that sub-period. The inventory rationing policy depends on one static

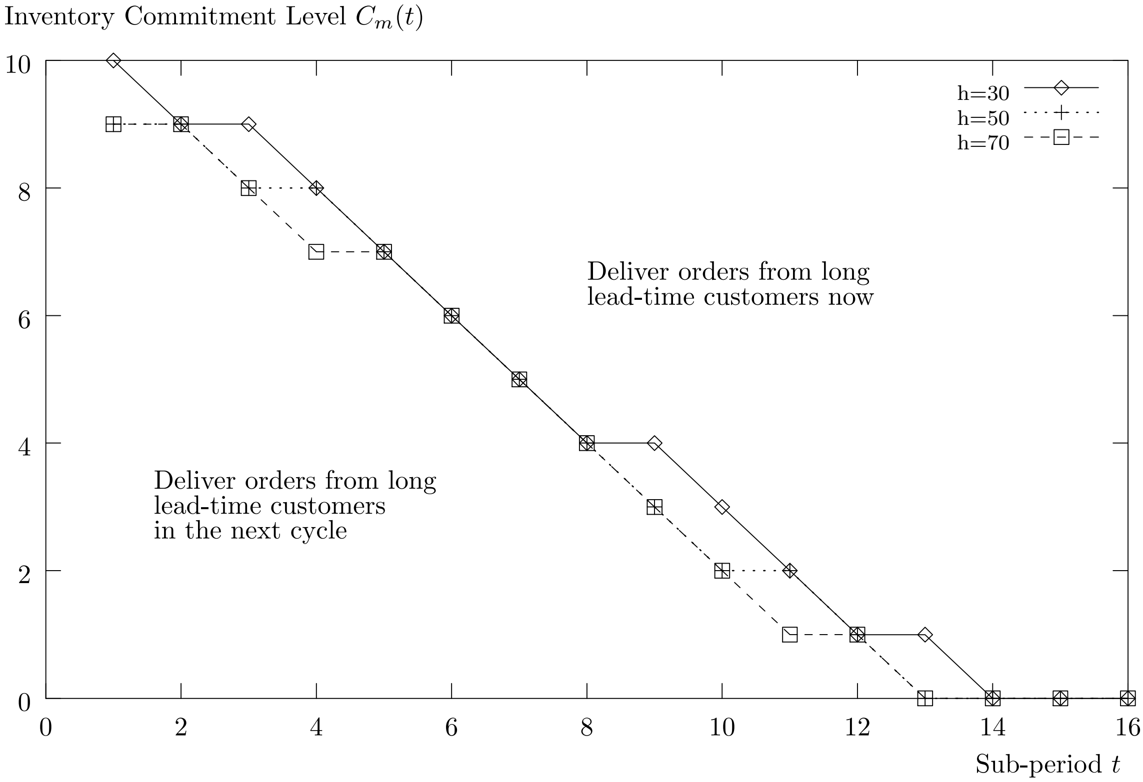

Figure 3: Illustration of dynamic inventory-commitment levels for different inventory holding cost h

inventory rationing level. With prevailing probabilities of different customers in Table 9, we calculate profits of the supplier under dynamic and static policies. Assume $p_l = \alpha p_s$ such that $\alpha \leq 1$. This is because the long lead-time products are inferior to the short lead-time products for customers. We use α as the price decline rate to characterize this inferiority. Let $\alpha = 0.95$.

Firstly we obtain the optimal inventory-commitment levels in Figure 3, which are nonincreasing as the time elpases. We obtain the optimal profit for the supplier with respect to inventory holding cost in Table 1. Note that the optimal static inventory rationing policy is in the sense that it chooses a best static inventory rationing level; and the improvement ratio equals to the profit difference of the two policies divided by the profit of the static one. From Table 1 we know that the optimal profit for the dynamic inventory-commitment policy is greater than that of the static inventory rationing policy, especially when the inventory holding cost is low.

Further, we shorten the horizon length to be 8, and obtain that the profits for the dynamic inventory-commitment policy and the optimal static inventory rationing policy are 193.91 and

Table 1: Profit Comparison for Dynamic Inventory Commitment and Static Inventory Rationing Policy

Inventory holding cost h	Optimal Operational Profit		Improvement Ratio(%)
	Static Rationing Policy	Dynamic Commitment Policy	
70	307.34	319.80	4.1%
60	308.09	321.12	4.2%
50	308.84	322.45	4.4%
40	309.59	324.26	4.7%
30	310.34	326.87	5.3%

190.24, respectively. And the improvement ratio is 1.9%, which is smaller than the improve ratio for the model whose horizon length is 16.

5.1 The robustness of the inventory-commitment policy

From Figure 3, we can see that the optimal inventory-commitment level in each sub-period doesn't change significantly for different inventory holding cost h. That is, this policy is robust to changes of the inventory holding cost. We claim that the supplier can use the dynamic inventory-commitment policy, even for the cases when the value of the inventory holding cost isn't very clear. Now we carry out a simple numerical experiment to verify this claim. From Table 1, we know that, when the inventory holding cost $h = 60$, the optimal operational profits for static and dynamic policy are 308.09 and 321.12, respectively. When the inventory holding cost is $h = 60$, and the supplier uses the optimal dynamic inventory commitment policies for $h = 70$ and $h = 50$, we can calculate the supplier's operational profits to be 320.65 and 321.10, respectively, both of which are very close to the optimal operational profit "321.12". This demonstrates that the optimal inventory-commitment levels are robust to the changes of the inventory holding cost.

For the relationship with other cost parameters, we have the following lemma. Its proof appears in Appendix.

Lemma 5.1 *The optimal inventory-commitment level is independent from the unit purchasing cost c, the unit salvage value s, the short lead-time price p_s and the long lead-time price p_f.*

This lemma gives much confidence for managers to use this inventory commitment policy, especially when the parameters are not exactly known in practice.

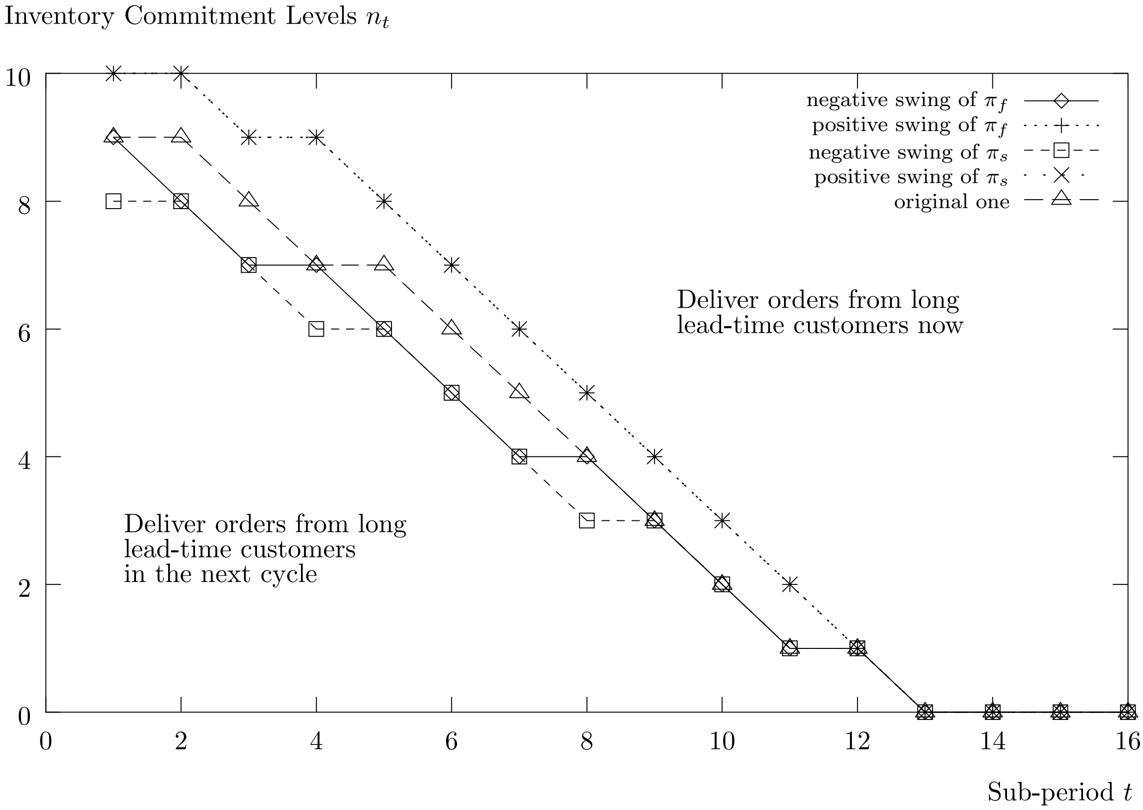

Figure 4: Illustration of dynamic inventory-commitment levels for different probabilities

Table 2: Profits for Different Policies

Different Policies	fix π_s, change π_f negative swing	fix π_s, change π_f positive swing	fix π_f, change π_s negative swing	fix π_f, change π_s positive swing	optimal
Profit	319.22	315.20	318.13	315.20	319.80
Change Rate	-0.2%	-1.4%	-0.5%	-1.4%	-

Further, we change the probabilities in sub-periods: fix π_s and let π_f of Table 9 in each sub-period have a negative (positive) swing of $\max\{0.05, \pi_f\}$ ($\min\{0.05, 1-\pi_s\}$); fix π_f and let π_s of Table 9 in each sub-period have a negative (positive) swing of $\max\{0.05, \pi_s\}$ ($\min\{0.05, 1-\pi_f\}$). Let $\alpha = 0.95$ and $h = 70$. Then we compare the inventory-commitment levels for negative/positive swings of long lead-time demand model, negative/positive swings of short lead-time demand model and the original one in Figure 4; and the operational profit for these inventory-commitment policies in Table 2, where the change rate equals to the profit difference with the optimal profit divided by the optimal profit. This demonstrates that both the optimal inventory-commitment levels and profits are robust to the changes of probabilities in sub-periods.

6 Benefit of Choice Model: Risk Pooling Effect

In order to characterize the benefits and pitfalls of providing lead-time choices to customers, in this section, we compare models with different delivery policies. In the first model, the non-choice model, short and long lead-time customers are satisfied by separately. In the second model, the choice model, short and long lead-time customers are satisfied jointly, and the optimal inventory-commitment policy, developed in Section 3 (Theorem 3.1), is deployed. In the choice model, by pooling long lead-time customers together, the risk from demand uncertainty of short lead-time customers is compansated. In what folllows, we develop the detailed models and use the profit difference of these two models to demonstrate the benefits of the choice model.

In the non-choice model, short and long lead-time customers are satisfied separately by on-hand inventory and replenished inventory of the next cycle, respectively. The leftover inventory of the short lead-time supplier are salvaged at the end of the cycle. The long lead-time supplier satisfies all the orders from long lead-time customers. The total profit, denoted by $V_{si}(n(1))$, can be written as

$$V_{si}(n(1)) \quad = \quad -c * n(1) + V_s(1, n(1)) + V_l, \tag{13}$$

where

$$\begin{aligned} V_s(1, n(1)) &= E_{D_s}[p_s \min\{\sum_{t=1}^{T} D_s(t), n(1)\} + (-h + s) \max\{n(1) - \sum_{t=1}^{T} D_s(t), 0\}], \\ V_l &= (p_f - c) E_{D_l}[\sum_{t=1}^{T} D_l(t)]. \end{aligned} \tag{14}$$

$-c * n(1)$ is the purchasing cost at the beginning of the current cycle. $V_s(1, n(1))$ is the profit for short lead-time customers, which are satisfied by the initial on-hand inventory $n(1)$ of the current cycle. This is similar to the traditional newsvendor model. V_l is the profit for long lead-time customers, all of which are satisfied by the replenished inventory of the next cycle.

In the choice model, short and long lead-time customers are satisfied jointly. The profit can be calculated by Equation (3), (4) and (5), and the optimal inventory-commitment policy and optimal inventory replenishment policy are characterized by Theorem 3.1 and 4.1, respectively. Since the inventory delivery policy for the non-choice model is not the optimal one, we can obtain that the profit of the choice model is no less than the non-choice model.

Table 3: Probabilites of Coming Customers in Sub-periods

Sub-period	1	2	3	4	5	6	7	8
π_0	0.46	0.35	0.26	0.19	0.14	0.11	0.1	0.11
$\pi_s = \pi_f$	0.27	0.325	0.37	0.405	0.43	0.445	0.45	0.445
Sub-period	9	10	11	12	13	14	15	16
π_0	0.14	0.19	0.26	0.35	0.46	0.59	0.74	0.91
$\pi_s = \pi_f$	0.43	0.405	0.37	0.325	0.27	0.205	0.13	0.045

6.1 Comparisons of two models

Assume that the unit purchasing price c is 70. The initial inventory level $x = 0$. The unit holding cost $h = 70$ and the slavage value $s = 20$. One cycle includes 16 sub-periods, i.e. $T = 16$. In sub-period t, assume that the probability for no customer $\pi_0(t) = \frac{(t-7)^2}{100} + 0.1$, and the prabablities for short and long lead-time customers are equal, i.e. $\pi_s(t) = \pi_f(t) = \frac{1-\pi_0}{2}$. Probabilities for no customer, short and long lead-time customers are shown in Table 3. The unit price of the short lead-time products p_s is 100, and the unit price of the long lead-time products $p_l = \alpha p_s$, with $\alpha \leq 1$. We compare profits between non-choice and choice models with a different price decline rate α.

In order to facilitate the numerical experiment for the non-choice model, according to the supplier's policy in the non-choice model, we rewrite $V_s(1, n(1))$ in a recursive form, for $n \geq 1$,

$$\begin{aligned} V_s(t,n) &= (1-\pi_s)V_s(t+1,n) + \pi_s(p_s + V_s(t+1,n-1)), \text{for } t < T; \\ V_s(T,n) &= (-h+s)n. \end{aligned} \tag{15}$$

And $V_s(t, 0) = 0$. Then we obtain the optimal profit $V_{si}^* = \max_{n(1)}\{V_{si}(n(1))\} = -c * n(1) + V_s(1, n(1)) + V_l$ for the non-choice model. For the choice model, we use expressions (3), (4) and (5) to calculate the optimal profit. These results are listed and compared in Table 4. Note that the "Improvement Ratio" is the profit difference divided by the profit of the non-choice model.

Moreover, we fix the price decline rate $\alpha = 0.95$, and calculate the optimal profit for different inventory holding cost h in Table 5.

If we use the improvement ratio as an indicator for risk-pooling effect, from Table 4, it indicates that the larger the difference in prices is, the higher the improvement ratio is, and from Table 5, it indicates that the higher the inventory holding cost is, the higher the improvement ratio is.

Table 4: Profits for Different Price Decline Rate, $h = 70$

Price Decline Rate	Optimal Operational Profit		Improvement
α	Non-choice Model	Choice Model	Ratio (%)
1.00	245.56	292.83	19.3%
0.98	234.92	282.19	20.1%
0.95	218.96	264.36	20.7%
0.90	192.36	239.63	24.6%
0.80	139.16	186.43	34.0%

Table 5: Profits for Different Inventory Holding Cost, $\alpha = 0.95$

Inventory Holding Cost	Optimal Operational Profit		Improvement
h	Non-choice Model	Choice Model	Ratio (%)
70	218.96	266.23	21.6%
60	221.23	267.73	21.0%
50	223.49	269.35	20.5%
40	225.76	271.09	20.1%
30	228.03	273.10	19.8%

The above observation are all consist with our intuition.

6.2 Leftover inventory levels of two models

In order to further explore the risk-pooling effect, in this subsection, we use simulation to study the leftover inventory level at the end of the current cycle $n(T)$. Two models start with the same initial inventory level after ordering $n(1)$, and experience the same demand realization. We compare the expected value and the variance of leftover inventory levels.

Assume the price decline rate $\alpha = 0.95$. We use the probabilities in Table 3 to randomly generate 50 demand realizations. For the non-choice model, all the long lead-time customers are backlogged and committed to deliver in the next cycle. For the choice model, the long lead-time delivery follows the optimal inventory-commitment policy and the optimal inventory-commitment levels are obtained by Equations (3), (4) and (5), and listed in Table 6. Then we obtain the leftover

Table 6: The Switching Curve of the Optimal Inventory Commitment Policy for $\alpha = 0.95$

Sub-period t	1	2	3	4	5	6	7	8
$C_m(t)$	7	7	6	6	5	4	4	3
Sub-period	9	10	11	12	13	14	15	16
$C_m(t)$	2	1	1	0	0	0	0	0

Table 7: Leftover Inventory for Different Initial Inventories

Initial Inventory After Ordering $n(1)$	Leftover Inventory Level					
	Non-choice Model			Choice Model		
	Mean	SD	CV	Mean	SD	CV
15	9.44	4.47	0.47	2.88	3.57	1.24
10	4.84	3.93	0.81	0.74	2.11	2.85
8	3.44	3.23	0.94	0.42	1.61	3.83
5	1.74	2.00	1.15	0.20	0.98	4.90

inventory level and calculate the mean, the stadard deviation (SD) and the coefficient of variation (CV). The results are listed in Table 7. Note that CV is an alternative measure of variability, which equals to SD divided by the mean.

The data shows that, starting from the same initial inventory level, both the expected value and the standard deviation of the leftover inventory for the choice model are significantly smaller; and the coefficient of variation is greater in the choice model than in the non-choice model. That is, the choice model can effectively reduce the leftover inventory level, which saves the inventory holding cost, and increases the utilization of the on-hand inventory.

To end this section, we conclude the results as follows. (1) The optimal profit of the choice model is greater than the operational profit of the non-choice one. (2) The risk-pooling effects become stronger when the price difference and the inventory holding cost are high. (3) The risk-pooling effect of the choice model reduce both the mean and the standard deviation, but increase the coefficient of variation of the leftover inventory.

7 Benefit of Choice Model: Demand Induction and Cannibalization effects

We study the risk-pooling effect in the last section. Note that, results obtained in the last section are under an assumption that the demand doesn't change when the lead-time choices are presented to customers. However, one would expect that customers could change their order patterns when there is an alternative. Therefore, in this section, in order to further evaluate the benefits and pitfills of providing lead-time choice to customers, we consider the demand-induction and -cannibalization effects. We use the customer choice model (Novshek and Sonnenshein (1979)) to characterize these effects.

Specifically, we assume that the customer's maximum willingness-to-pay (W.T.P.) for products with current and next cycle delivery as w_s and w_l, respectively, where w_s and w_l depends on a joint distribution over R_+^2 of $g(w_s, w_l)$. R_+^2 is a two-dimention non-negative vector. And we assume that the maximum W.T.P. for the long lead-time delivery mode w_f is a linear function of the above two,

$$w_f(w_s, w_l) \quad = \quad \pi w_s + (1-\pi) w_l - \rho, \tag{16}$$

where the π is the customer's probability of receiving products at the current cycle and $\rho \geq 0$ is the premium in W.T.P. due to the delivery time uncertainty. We assume that $\pi = 0.5$ and $\rho = 3$.

For the non-choice model, the cusomer buys products as long as its W.T.P. is no less than the price of the product with a short delivery lead-time, i.e. $w_s \geq p_s$. Then the probability of the prevailing short lead-time customers is

$$\pi_s \quad = \quad \int_0^\infty \int_{p_s}^\infty g(w_s, w_l) dw_s dw_l. \tag{17}$$

Further assume that w_s and w_l are independently and uniformly distributed in $[0, W_s]$ and $[0, W_l]$, respectively. Then we can obtain that

$$\pi_s \quad = \quad \frac{W_s - p_s}{W_s}. \tag{18}$$

Let $p_s = 100$, $p_f = 95$ and $W_l = W_s$. To keep the probabilities π_0 the same as in Table 3, we set the value of W_s as in Table 8 according to Equation (18). In contract to the last section, we demonstrate how much additional profit can be obtained after including the demand induction and cannibalization into consideration. $\pi_s = 1 - \pi_0$. In this model, all short lead-time customers are satisfied once the supplier has positive on-hand inventory. We can use the same way as in Equation (15) to calculate the profit in Figure 5.

For the choice model, note that probabilities are different for these customers, which is due to demand-induction and -cannibalization effects. In what follows, we develop models for the demand-induction and -cannibalization.

Firstly, the demand-induction is defined as an extra demand is allured by a lower selling price. As it has been assumed, the price of the long lead-time products is no more than that of the short

Table 8: Probabilites of the Non-choice Model for Sub-periods

Sub-period	1	2	3	4	5	6	7	8
π_0	0.46	0.35	0.26	0.19	0.14	0.11	0.1	0.11
π_s	0.54	0.65	0.74	0.81	0.86	0.89	0.9	0.89
W_s	217.39	285.71	384.62	526.32	714.29	909.09	1000	909.09
Sub-period	9	10	11	12	13	14	15	16
π_0	0.14	0.19	0.26	0.35	0.46	0.59	0.74	0.91
π_s	0.86	0.81	0.74	0.65	0.54	0.41	0.26	0.09
W_s	714.29	526.32	384.62	285.71	217.39	169.49	135.14	109.89

lead-time one, e.g. $p_f = 95 \leq p_s = 100$. A demand induction takes place when the W.T.P. of a short lead-time customer is smaller than the prevailing short lead-time price p_s, and the W.T.P. of the long lead-time customer is greater than the prevailing long lead-time price p_f, i.e.

$$\begin{cases} w_s < p_s, \\ w_f \geq p_f; \end{cases} \Longleftrightarrow \begin{cases} w_s < p_s, \\ w_l \geq 2(\rho + p_f) - w_s. \end{cases} \tag{19}$$

where "$\Longleftrightarrow$" is by (16). Then we can obtain that the probability of the induced demand is

$$\pi_i = \int_0^{p_s} \int_{2(\rho+p_f)-w_s}^{+\infty} g(w_s, w_l) dw_l dw_s,$$

if w_s, w_l are indepently and uniformly distributed, then

$$\pi_i = \int_{max\{0,2(\rho+p_f)-W_l\}}^{p_s} \frac{W_l - 2(\rho + p_f) + w_s}{W_s W_l} dw_s. \tag{20}$$

From the above equation, we know that the induced demand π_i is increasing in the prevailing short lead-time price p_s, and decreasing in the prevailing long lead-time price p_f.

Secondly, the demand-cannibalization is defined as a short lead-time demand is taken away by the introduction of alternative lead-time product. Note that, for specific prices of two lead-time choices, customers are defined as the short lead-time type when they choose the short lead-time option. And the choice of these customers can change when the prices change. For example, when the price of the alternative long lead-time choice is modified to be low enough for some customers to compensate the penalty of unsatisfaction in the current cycle, these short lead-time customers will change to choose the long lead-time option. And they are long lead-time customers under the modified prices. The demand-cannibalization takes places when the W.T.P. of the short lead-time customer is greater than the prevailing short lead-time price p_s, and the margin (the W.T.P. w_f

minus the price p_f) of the long lead-time product is greater than the margin of the short lead-time product, i.e.

$$\begin{cases} w_s \geq p_s, \\ w_f - p_f \geq w_s - p_s; \end{cases} \Longleftrightarrow \begin{cases} w_s \geq p_s, \\ w_l \geq w_s - 2p_s + 2\rho + 2p_f. \end{cases} \tag{21}$$

where "$\Longleftrightarrow$" is by (16). Then we can obtain the probability of the cannibalized demand of the choice model is

$$\pi_c = \int_{p_s}^{+\infty} \int_{w_s - 2(p_s - p_f) + 2\rho}^{+\infty} g(w_s, w_l) dw_l dw_s,$$

if w_s, w_l are indepently and uniformly distributed, then

$$\pi_c = \int_{p_s}^{W_s} \frac{W_l - (-2p_s + 2\rho + 2p_f) - w_s}{W_s W_l} dw_s. \tag{22}$$

From the above equation, we know that the cannibalized demand π_c is decreasing in the prevailing long lead-time price p_f.

According to the analysis above, the probabilites can be different for the choice model. The appearing probability of the short lead-time customers is reduced by the cannibalization effect π_c; and the appearing probability of the long lead-time customers equals to the sum of the cannibalized and the induced ones, i.e.

- π_0' of the Choice Model = π_0 of the Non-choice Model - π_i
- π_s' of the Choice Model = π_s of the Non-choice Model - π_c
- π_f' of the Choice Model = π_i + π_c

By Equations (20) and (22), we obtain the probabilities for each sub-period in Table 9.

Then we calculate the profit for the choice model and compare these two models in Figure 5. We see that the profit is significantly increasing when the long lead-time choice is included, even with the price decline. And the *optimal* profit increases by $(319.80 - 249.62)/249.62 = 28.1\%$. Moreover, we compare the maximal profits between the non-choice and choice models with different price decline rate α (Remind that $p_f = \alpha * p_s$) when $h = 70$ and $h = 50$ in Table 10 and 11, respectively; and with different inventory holding cost when $\alpha = 0.95$ in Table 12. From the data, we know that the profit improvement of introducing long lead-time deliver option

Table 9: Probabilites of the Choice Model for Sub-periods

Sub-period	1	2	3	4	5	6	7	8
π_i	0.15	0.17	0.16	0.14	0.11	0.09	0.09	0.09
π_c	0.16	0.22	0.28	0.33	0.37	0.4	0.4	0.4
π'_0	0.31	0.18	0.10	0.05	0.03	0.02	0.01	0.02
π'_s	0.38	0.43	0.46	0.48	0.49	0.49	0.50	0.49
π'_f	0.31	0.39	0.44	0.47	0.48	0.49	0.49	0.49
Sub-period	9	10	11	12	13	14	15	16
π_i	0.11	0.14	0.16	0.17	0.15	0.09	0.04	0.01
π_c	0.37	0.33	0.28	0.22	0.16	0.09	0.04	0.01
π'_0	0.03	0.05	0.10	0.18	0.31	0.50	0.70	0.90
π'_s	0.49	0.48	0.46	0.43	0.38	0.32	0.22	0.08
π'_f	0.48	0.47	0.44	0.39	0.31	0.18	0.08	0.02

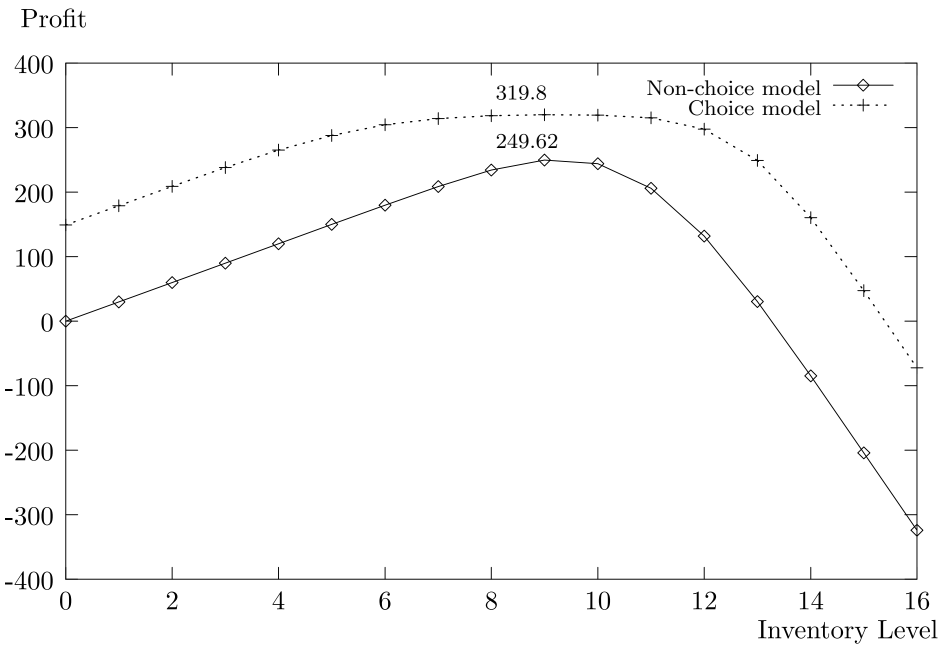

Figure 5: Profit for different inventory after replenishment when $h = 70$ and $\alpha = 0.95$

Table 10: Profit for Different Price Decline Rate, $h = 70$

Price Decline Rate	Optimal Profit		Improvement
α	Non-choice Model	Choice Model	Ratio (%)
1.00		349.61	40.1%
0.98		337.69	35.3%
0.95	249.62	319.80	28.1%
0.90		289.99	16.2%
0.83		248.25	−0.5%
0.80		230.36	−7.7%

Table 11: Profit for Different Price Decline Rate, $h = 50$

Price Decline Rate	Optimal Profit		Improvement
α	Non-choice Model	Choice Model	Ratio (%)
1.00		352.26	39.6%
0.98		340.34	34.9%
0.95	252.34	322.45	27.8%
0.90		292.64	16.0%
0.83		250.90	−0.6%
0.80		233.02	−7.7%

can be significant, and the greater the price decline rate (the larger the inventory holding cost), the greater the improvement ratio. However, when the price decline is significant, introducing long lead-time option can be questionable. The profit loss is due to the cannibalized demand from short to long lead-time customers and the significant unit price decline for long lead-time customers.

Table 12: Profit for Different Inventory Holding Cost, $\alpha = 0.95$

Inventory Holding Cost	Optimal Profit		Improvement
h	Non-choie Model	Choice Model	Ratio (%)
70	249.62	319.80	28.1%
60	250.98	321.12	27.9%
50	252.34	322.45	27.8%
40	255.22	324.26	27.1%
30	258.95	326.87	26.2%

8 Concluding Remarks and Further Research Direction

This paper introduces the alternative lead-time choices for the customers into the traditional fixed and unique lead-time inventory model, and characterizes the optimal inventory replenishment policy, as well as the optimal inventory-commitment policy within the cycle. Compared with the production lead-time of the supplier, the lead-time choices for the customers are divided into two catagories of the short and the long lead-time requirement: the former one asks for a delivery lead-time which is shorter than the production lead-time of the supplier, and the latter one asks for a delivery lead-time which is longer than the production lead-time of the supplier.

This paper demonstrates the profit improvement by including alternative lead-time choices with both analytical results and numerical experiments. The improved profit comes from the following sources: (1) The demand-induction effect. The lower price of the long lead-time choice attracts the additional customers; (2) The risk-pooling effect. The long lead-time customers provide a flexibility to the supplier in deciding when to deliver the products. The supplier balance the trade-off between the following two things: delivering products in the next cycle to save on-hand inventory, which meets potential short lead-time orders in the remaining cycle, and delivering on-hand inventory now to long lead-time customers in inventory holding cost reduction. This enables the supplier to use the future capacity together with the current capacity to satisfy the demand. And this also enables the supplier to better control the on-hand inventory during the selling cycle.

Our results also demonstrate that the risk-pooling effect is increasing in both the price difference of short and long lead-time customers and the inventory holding cost. The demand induction and cannibaliztion effect is increasing in both price decline rate and the inventory holding cost.

Note that the risk pooling, demand-induction and -cannibalization effects of flexible products in a revenue management model are investigated by Gallego and Phillips (2004). The concept of flexible products is similar to that of the long lead-time products, because both of them let the supplier choose from a feasible set, e.g. two feasible products or two feasible delivery lead-times. But the concept of the long lead-time products in our paper focuses on the flexibility of deliver time, rather than the flexibility of what to deliver. And the supplier has an opportunity of inventory replenishment when it chooses to deliver in the next cycle. We focus on both improving capacity utilization and reducing overall leftover inventory, and demonstrate that the risk-pooling benefit of choice model comes from a reduction of the mean and the variance of leftover inventory. The optimal inventory replenishment policy is also characterized in this paper.

Moreover, this paper compares supplier's profit between the optimal inventory-commitment policy and the static inventory rationing policy. Cattani and Souza (2002) compare the static inventory rationing policy with the first-come first-serve policy and study the conditions, under which the inventory rationing policy is beneficial. This paper demonstrates that the optimal inventory commitment policy out-performs the static inventory rationing policy, especially when the inventory holding cost is low and the cycle time is long. In addition, the optimal commitment level in each sub-period is independent from the unit purchasing cost, the unit salvage cost, the short and long lead-time prices, and robust to the inventory holding cost, so that the supplier can achieve a "close-to-optimal" profit even when he doesn't know these cost parameters exactly.

A very interesting issue for the future research is to introduce 3 or more lead-time choices into the choice model: long lead-time customers require a maximume delivery time which is greater than the production lead-time, medium lead-time customers require a maximume delivery time which is equal to the production lead-time and the short lead-time customers require a maximume delivery time which is shorter than the production lead-time. We need three-dimention vector (n, m_1, m_2) to describe the state during the selling cycle. n stands for the on-hand inventory plus the pipe-line inventory, m_1 and m_2 stand for the promised-to-deliver products in the next cycle and in the third cycle, respectively. Study of this three lead-time choice model will shed light to a general multiple lead-time choice model.

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A Proof of Lemma 3.1.

It is straightforward to check that $V(T, n-1, m) - V(T, n, m+1) = -c(m-n+1)^+ - hn + h + s(n-1-m)^+ + c(n+1-n) + hn - s(n-m-1)^+ = h$, which satisfies (a) and (b). Assume that $V(t+1, n-1, m) - V(t+1, n, m+1)$ satisfies (a) and (b). Then we need to prove that $V(t, n-1, m) - V(t, n, m+1)$ satisfies (a) and (b).

Proof of (a). After checking (4), we know that we only need to check the following inequality:

$$\begin{aligned} & \max\{V(t+1, n-2, m), V(t+1, n-1, m+1)\} \\ - & \max\{V(t+1, n-1, m+1), V(t+1, n, m+2)\} \\ \leq & \max\{V(t+1, n-1, m), V(t+1, n, m+1)\} \\ - & \max\{V(t+1, n, m+1), V(t+1, n+1, m+2)\}. \end{aligned} \tag{23}$$

We check different cases in the remaining proof.

$$\begin{aligned} & V(t+1, n-2, m) - \max\{V(t+1, n-1, m+1), V(t+1, n, m+2)\} \\ \leq & V(t+1, n-2, m) - V(t+1, n-1, m+1) \end{aligned}$$

$$
\begin{aligned}
&\leq V(t+1,n-1,m) - V(t+1,n,m+1) \\
&\leq \max\{V(t+1,n-1,m), V(t+1,n,m+1)\} - V(t+1,n,m+1), \qquad (24)
\end{aligned}
$$

where the second inequality is by that $V(t+1,n-1,m) - V(t+1,n,m+1)$ satisfies (a).

$$
\begin{aligned}
& V(t+1,n-2,m) - \max\{V(t+1,n-1,m+1), V(t+1,n,m+2)\} \\
\leq\ & V(t+1,n-2,m) - V(t+1,n-1,m+1) \\
\leq\ & V(t+1,n-1,m+1) - V(t+1,n,m+2) \\
\leq\ & V(t+1,n,m+1) - V(t+1,n+1,m+2) \\
\leq\ & \max\{V(t+1,n-1,m), V(t+1,n,m+1)\} - V(t+1,n+1,m+2), \qquad (25)
\end{aligned}
$$

where the second and third inequalities is by that $V(t+1,n-1,m) - V(t+1,n,m+1)$ satisfies (b) and (a), respectively.

$$
\begin{aligned}
& V(t+1,n-1,m+1) - \max\{V(t+1,n-1,m+1), V(t+1,n,m+2)\} \\
\leq\ & 0 \\
\leq\ & \max\{V(t+1,n-1,m), V(t+1,n,m+1)\} - V(t+1,n,m+1). \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
& V(t+1,n-1,m+1) - \max\{V(t+1,n-1,m+1), V(t+1,n,m+2)\} \\
\leq\ & V(t+1,n-1,m+1) - V(t+1,n,m+2) \\
\leq\ & V(t+1,n,m+1) - V(t+1,n+1,m+2) \\
\leq\ & \max\{V(t+1,n-1,m), V(t+1,n,m+1)\} - V(t+1,n+1,m+2), \qquad (27)
\end{aligned}
$$

where the second inequality is by that $V(t+1,n-1,m+1) - V(t+1,n,m+2)$ satisfies (a).

Combine the above four inequalities we can obtain (23).

Proof of (b). After checking (4), we know that we only need to check the following equality:

$$
\begin{aligned}
& \max\{V(t+1,n-2,m), V(t+1,n-1,m+1)\} \\
-\ & \max\{V(t+1,n-1,m+1), V(t+1,n,m+2)\} \\
=\ & \max\{V(t+1,n-2,m+1), V(t+1,n-1,m+2)\} \\
-\ & \max\{V(t+1,n-1,m+2), V(t+1,n,m+3)\}. \qquad (28)
\end{aligned}
$$

We check different cases in the remaining proof.

When $V(t+1,n-2,m) \geq V(t+1,n-1,m+1)$ and $V(t+1,n-1,m+1) \geq V(t+1,n,m+2)$, by $V(t+1,n-1,m) - V(t+1,n,m+1)$ satisfies (b) we know that $V(t+1,n-2,m+1) \geq V(t+1,n-1,m+2)$ and $V(t+1,n-1,m+2) \geq V(t+1,n,m+3)$. Hence (28) is equivalent to

$$\begin{aligned} & V(t+1,n-2,m) - V(t+1,n-1,m+1) \\ = \; & V(t+1,n-2,m+1) - V(t+1,n-1,m+2), \end{aligned} \tag{29}$$

which is true by that $V(t+1,n-2,m) - V(t+1,n-1,m+1)$ satisfies (b).

When $V(t+1,n-2,m) \geq V(t+1,n-1,m+1)$ and $V(t+1,n-1,m+1) \leq V(t+1,n,m+2)$, by $V(t+1,n-1,m) - V(t+1,n,m+1)$ satisfies (b) we know that $V(t+1,n-2,m+1) \geq V(t+1,n-1,m+2)$ and $V(t+1,n-1,m+2) \leq V(t+1,n,m+3)$. Hence (28) is equivalent to

$$\begin{aligned} & V(t+1,n-2,m) - V(t+1,n,m+2) \\ = \; & V(t+1,n-2,m+1) - V(t+1,n,m+3). \end{aligned} \tag{30}$$

Since $V(t+1,n-2,m) - V(t+1,n-1,m+1)$ satisfies (b), we have $V(t+1,n-2,m) - V(t+1,n,m+2) = V(t+1,n-2,m) - V(t+1,n-1,m+1) + V(t+1,n-1,m+1) - V(t+1,n,m+2) = V(t+1,n-2,m+1) - V(t+1,n-1,m+2) + V(t+1,n-1,m+2) - V(t+1,n,m+3) = V(t+1,n-2,m+1) - V(t+1,n,m+3)$.

When $V(t+1,n-2,m) \leq V(t+1,n-1,m+1)$ and $V(t+1,n-1,m+1) \geq V(t+1,n,m+2)$, by $V(t+1,n-1,m) - V(t+1,n,m+1)$ satisfies (b) we know that $V(t+1,n-2,m+1) \leq V(t+1,n-1,m+2)$ and $V(t+1,n-1,m+2) \geq V(t+1,n,m+3)$. Hence (28) is equivalent to

$$\begin{aligned} & V(t+1,n-1,m+1) - V(t+1,n-1,m+1) \\ = \; & V(t+1,n-1,m+2) - V(t+1,n-1,m+2), \end{aligned} \tag{31}$$

which is obviously true.

When $V(t+1,n-2,m) \leq V(t+1,n-1,m+1)$ and $V(t+1,n-1,m+1) \leq V(t+1,n,m+2)$, by $V(t+1,n-1,m) - V(t+1,n,m+1)$ satisfies (b) we know that $V(t+1,n-2,m+1) \leq V(t+1,n-1,m+2)$ and $V(t+1,n-1,m+2) \leq V(t+1,n,m+3)$. Hence (28) is equivalent to

$$\begin{aligned} & V(t+1,n-1,m+1) - V(t+1,n,m+2) \\ = \; & V(t+1,n-1,m+2) - V(t+1,n,m+3), \end{aligned} \tag{32}$$

which is true by that $V(t+1, n-1, m+1) - V(t+1, n, m+2)$ satisfies (b).

Combine the above four equalities we can obtain (28). $\square$

B Proof of Theorem 3.1.

According to Equation (4), it is optimal to deliver now if $V(t+1, n-1, m) - V(t+1, n, m+1) \geq 0$; and optimal to deliver in the next cycle, otherwise. Lemma 3.1 provides that, for a specific sub-period t, the value of $V(t+1, n-1, m) - V(t+1, n, m+1)$ is non-decreasing in n and independent from m. Hence, if $V(t+1, 0, m) - V(t+1, 1, m+1) \geq 0$, then $V(t+1, n-1, m) - V(t+1, n, m+1) \geq 0$ for all $n \geq 1$. In this case, $C_m(t) = 0$ and the optimal policy is to deliver orders from long lead-time customers now for all $n \geq 1$. Otherwise, $C_m(t)$ is the largest value of n such that $V(t+1, n-1, m) - V(t+1, n, m) \leq 0$. Obviously $C_m(t)$ is independent from m.

Following is the proof of the non-increasing property of $C_m(t)$, with respect to sub-period t. According to the definition of $C_m(t)$, we know that

$$\begin{aligned} V(t+1, C_m(t)-1, m) &\leq V(t+1, C_m(t), m+1) \\ V(t+1, C_m(t)-2, m) &\leq V(t+1, C_m(t)-1, m+1). \end{aligned} \tag{33}$$

By (4), we obtain

$$\begin{aligned} & V(t, C_m(t)-1, m) - V(t, C_m(t), m+1) \\ = \ & \pi_0[V(t+1, C_m(t)-1, m) - V(t+1, C_m(t), m+1)] \\ + \ & \pi_f[V(t+1, C_m(t)-1, m+1) - V(t+1, C_m(t), m+2)] \\ + \ & \pi_s[V(t+1, C_m(t)-2, m) - V(t+1, C_m(t)-1, m+1)]. \end{aligned} \tag{34}$$

By (33) and Lemma 3.1, we have $V(t+1, C_m(t)-1, m) - V(t+1, C_m(t), m+1) = V(t+1, C_m(t)-1, m+1) - V(t+1, C_m(t), m+2) \leq 0$ and $V(t+1, C_m(t)-2, m) - V(t+1, C_m(t)-1, m+1) \leq 0$. Hence we obtain

$$V(t, C_m(t)-1, m) - V(t, C_m(t), m) \leq 0. \tag{35}$$

Since $C_m(t-1)$ is defined as the largest value n such that $V(t, n-1, m) - V(t, n, m) \leq 0$. Then we obtain that $C_m(t-1) \geq C_m(t)$. $\square$

C Proof of Lemma 4.1

First of all, it is straightforward that $V(T,n,m) \in \hat{V}$. Now we assume that $V(t+1,n,m) \in \hat{V}$ and prove that $V(t,n,m) \in \hat{V}$. From the equation of $V(t,n,m)$, we know that we only need to focus on the part of $\max\{V(t+1,n-1,m), V(t+1,n,m+1)\}$. In the following we prove that

$$T(n,m) := \max\{V(t+1,n-1,m), V(t+1,n,m+1)\}$$

satisfies P1, P2, P3 step by step.

Proof of P1. By Lemma 3.1, we know that given m, there exists a $C_m(t)(m)$ such that for $n \leq C_m(t)$, $T(n,m) = V(t+1,n,m+1)$ is concave in n; and for $n \geq C_m(t)+1$, $T(n,m) = V(t+1,n-1,m)$ is concave in n. We now only need to obtain the following inequalities to prove the concavity of $T(n,m)$ in n: (In the following of the part Proof of P1, without confusion we use $T(n)$ instead of $T(n,m)$ for notation simplification)

$$T(C_m(t)+2) - T(C_m(t)+1) \leq T(C_m(t)+1) - T(C_m(t)) \tag{36}$$

$$T(C_m(t)+1) - T(C_m(t)) \leq T(C_m(t)) - T(C_m(t)-1) \tag{37}$$

or equivalently,

$$V(t+1,n_t+1,m) - V(t+1,C_m(t),m) \leq V(t+1,C_m(t),m) - V(t+1,C_m(t),m+1) \tag{38}$$

$$V(t+1,C_m(t),m) - V(t+1,C_m(t),m+1) \leq V(t+1,C_m(t),m+1) - V(t+1,C_m(t)-1,m+1). \tag{39}$$

We have

$$V(t+1,C_m(t),m+1) \geq V(t+1,C_m(t)-1,m) \tag{40}$$

$$V(t+1,C_m(t),m) \geq V(t+1,C_m(t)+1,m+1) \tag{41}$$

Then we can obtain that

$$\begin{aligned} & V(t+1,C_m(t),m) - V(t+1,C_m(t),m+1) \\ \leq\; & V(t+1,C_m(t),m) - V(t+1,C_m(t)-1,m) \\ \leq\; & V(t+1,C_m(t),m+1) - V(t+1,C_m(t)-1,m+1), \end{aligned} \tag{42}$$

where the first inequality is by (40), and the second one is by P3 of $V(t+1, C_m(t), m)$. Hence we obtain (39).

We also can obtain that

$$\begin{aligned} & V(t+1, C_m(t), m) - V(t+1, C_m(t), m+1) \\ \geq \ & V(t+1, C_m(t)+1, m+1) - V(t+1, C_m(t), m+1) \\ \geq \ & V(t+1, n_{t+1}+1, m) - V(t+1, C_m(t), m), \end{aligned} \tag{43}$$

where the first inequality is by (41) and the second one is by P3 of $V(t+1, C_m(t)+1, m)$. Hence we obtain (38).

Proof of P2. By Lemma 3.1, we know that given n, there exists a $m_{t+1}(n)$ such that for $m \leq m_{t+1}$, $T(n,m) = V(t+1, n, m+1)$ is concave in m; and for $m \geq m_{t+1}$, $T(n,m) = V(t+1, n-1, m)$ is concave in m. We now only need to obtain the following inequalities to prove the concavity of $T(n,m)$ in m: (In the following of the part Proof of P2, without confusion we use $T(m)$ instead of $T(n,m)$ for notation simplification)

$$T(m_{t+1}+2) - T(m_{t+1}+1) \leq T(m_{t+1}+1) - T(m_{t+1}) \tag{44}$$

$$T(m_{t+1}+1) - T(m_{t+1}) \leq T(m_{t+1}) - T(m_{t+1}-1) \tag{45}$$

or equivalently,

$$\begin{aligned} & V(t+1, n-1, m_{t+1}+2) - V(t+1, n-1, m_{t+1}+1) \\ \leq \ & V(t+1, n-1, m_{t+1}+1) - V(t+1, n, m_{t+1}+1); \end{aligned} \tag{46}$$

$$\begin{aligned} & V(t+1, n-1, m_{t+1}+1) - V(t+1, n, m_{t+1}+1) \\ \leq \ & V(t+1, n, m_{t+1}+1) - V(t+1, n, m_{t+1}). \end{aligned} \tag{47}$$

We have

$$V(t+1, n-1, m_{t+1}+1) \geq V(t+1, n, m_{t+1}+2) \tag{48}$$

$$V(t+1, n-1, m_{t+1}) \leq V(t+1, n, m_{t+1}+1) \tag{49}$$

Then we can obtain that

$$V(t+1, n, m_{t+1}+1) - V(t+1, n, m_{t+1})$$

$$
\begin{aligned}
&\geq\ V(t+1,n-1,m_{t+1}) - V(t+1,n,m_{t+1}) \\
&\geq\ V(t+1,n-1,m_{t+1}+1) - V(t+1,n,m_{t+1}+1), \qquad (50)
\end{aligned}
$$

where the first inequality is by (49), and the second one is by P3 of $V(t+1,n,m_{t+1}+1)$. Hence we obtain (47).

We also can obtain that

$$
\begin{aligned}
&V(t+1,n-1,m_{t+1}+1) - V(t+1,n,m_{t+1}+1) \\
\geq\ &V(t+1,n,m_{t+1}+2) - V(t+1,n,m_{t+1}+1) \\
\geq\ &V(t+1,n-1,m_{t+1}+2) - V(t+1,n-1,m_{t+1}+1), \qquad (51)
\end{aligned}
$$

where the first inequality is by (48) and the second one is by P3 of $V(t+1,n,m_{t+1}+2)$. Hence we obtain (46).

Proof of P3. Similarly, by equation (4), we only need to show that $T(n,m)$ satisfies P3, i.e.

$$
\begin{aligned}
&\max\{V(t+1,n-1,m), V(t+1,n,m+1)\} \\
-\ &\max\{V(t+1,n-2,m), V(t+1,n-1,m-1)\} \\
\leq\ &\max\{V(t+1,n-1,m+1), V(t+1,n,m+2)\} \\
-\ &\max\{V(t+1,n-2,m+1), V(t+1,n-1,m+2)\} \qquad (52)
\end{aligned}
$$

In the following, we check different cases.

$$
\begin{aligned}
&V(t+1,n-1,m) - \max\{V(t+1,n-2,m), V(t+1,n-1,m+1)\} \\
\leq\ &V(t+1,n-1,m) - V(t+1,n-2,m) \\
\leq\ &V(t+1,n-1,m+1) - V(t+1,n-2,m+1) \\
\leq\ &\max\{V(t+1,n-1,m+1), V(t+1,n,m+2)\} - V(t+1,n-2,m+1), \qquad (53)
\end{aligned}
$$

where the second inequality is by P3 of $V(t+1,n,m)$.

$$
\begin{aligned}
&V(t+1,n-1,m) - \max\{V(t+1,n-2,m), V(t+1,n-1,m+1)\} \\
\leq\ &V(t+1,n-1,m) - V(t+1,n-1,m+1) \\
\leq\ &V(t+1,n-1,m+1) - V(t+1,n-1,m+2) \\
\leq\ &\max\{V(t+1,n-1,m+1), V(t+1,n,m+2)\} - V(t+1,n-1,m+2), \qquad (54)
\end{aligned}
$$

where the second inequality is by P2 of $V(t+1,n-1,m)$.

$$\begin{aligned} & V(t+1,n,m+1) - \max\{V(t+1,n-2,m), V(t+1,n-1,m+1)\} \\ \leq\ & V(t+1,n,m+1) - V(t+1,n-1,m+1) \\ \leq\ & V(t+1,n,m+2) - V(t+1,n-1,m+2) \\ \leq\ & \max\{V(t+1,n-1,m+1), V(t+1,n,m+2)\} - V(t+1,n-1,m+2), \end{aligned} \tag{55}$$

where the second inequality is by P3 of $V(t+1,n,m)$.

$$\begin{aligned} & V(t+1,n,m+1) - \max\{V(t+1,n-2,m), V(t+1,n-1,m+1)\} \\ \leq\ & V(t+1,n,m+1) - V(t+1,n-1,m+1) \\ \leq\ & V(t+1,n-1,m+1) - V(t+1,n-2,m+1) \\ \leq\ & \max\{V(t+1,n-1,m+1), V(t+1,n,m+2)\} - V(t+1,n-2,m+1), \end{aligned} \tag{56}$$

where the second inequality is by P1 of $V(t+1,n,m)$. And combine the above 4 inequalities we obtain the inequality (52). □

D Proof of Lemma 5.1

It is straightforward to check that $V(T,n-1,m) - V(T,n,m+1) = -c(m-n+1)^+ - hn + h + s(n-1-m)^+ + c(n+1-n) + hn - s(n-m-1)^+ = h$, which is independent from c, s, p_s and p_f. Assume that $V(t+1,n-1,m) - V(t+1,n,m+1)$ is independent from c, s, p_s and p_f. Then we need to prove that $V(t,n-1,m) - V(t,n,m+1)$ is independent from c, s, p_s and p_f.

After checking (4), we know that we only need to check

$$\begin{aligned} & \max\{V(t+1,n-2,m), V(t+1,n-1,m+1)\} \\ & -\max\{V(t+1,n-1,m+1), V(t+1,n,m+2)\} \end{aligned} \tag{57}$$

is independent from p_s and p_f. We check different cases in the remaining proof.

When $V(t+1,n-2,m) \geq V(t+1,n-1,m+1)$ and $V(t+1,n-1,m+1) \geq V(t+1,n,m+2)$, Equation (57) is equivalent to $V(t+1,n-2,m) - V(t+1,n-1,m+1)$, which is independent from c, s, p_s and p_f.

When $V(t+1,n-2,m) \geq V(t+1,n-1,m+1)$ and $V(t+1,n-1,m+1) \leq V(t+1,n,m+2)$, Equation (57) is equivalent to $V(t+1,n-2,m)-V(t+1,n,m+2) = [V(t+1,n-2,m)-V(t+1,n-1,m+1)]+[V(t+1,n-1,m+1)-V(t+1,n,m+2)]$. Since $V(t+1,n-2,m)-V(t+1,n-1,m+1)$ and $V(t+1,n-1,m+1)-V(t+1,n,m+2)$ are all independent from c, s, p_s and p_f, then Equation (57) is independent from c, s, p_s and p_f.

When $V(t+1,n-2,m) \leq V(t+1,n-1,m+1)$ and $V(t+1,n-1,m+1) \geq V(t+1,n,m+2)$, Equation (57) is equivalent to 0, which is independent from c, s, p_s and p_f.

When $V(t+1,n-2,m) \leq V(t+1,n-1,m+1)$ and $V(t+1,n-1,m+1) \leq V(t+1,n,m+2)$, Equation (57) is equivalent to $V(t+1,n-1,m) - V(t+1,n,m+1)$, which is independent from c, s, p_s and p_f. □