Optimal Inventory Decisions in the Multi-period News-vendor Problem with Partial-Observed Markovian Supply Capacities

Haifeng Wang, and Houmin Yan

Abstract—This paper considers a multi-period news-vendor problem with partially observed supply capacity information which evolves as a Markovian Process. The supply capacity is fully observed by the buyer when the capacity is smaller than buyer’s ordering quantity. With a dynamic programming formulation, we prove the existence of a unique optimal ordering policy. In a two-state Markovian capacity case, we further demonstrate that the buyer orders more than required to reveal supply capacity. We also provide a numerical example to demonstrate the characterization of the optimal policy.

Note to Practitioners—Supplier management is a critical task for inventory managers, in particular, when there existing supply uncertainties. In such circumstances, inventory managers have to make inventory replenishment decisions based on their best estimation of supply conditions. Their past experiences are the basis for estimation. Note that their past experiences, to the best, only partially reveals the supply condition. This paper develops a new model in which it makes use of a partial observed supply capacity information, which is derived in the order fulfillment process, to forecast the future supply capacity. There are two possible observations of the order fulfillment: (1) if the ordering quantity is greater than the current period supply capacity, the buyer observes the value of the current period supply capacity; (2) otherwise, the buyer knows that the current period supply capacity is greater than its ordering quantity. Based on these two observations, the buyer updates the future supply capacity forecasting accordingly. Compared with traditional models, we demonstrate that making use of the partial observed information significantly reduces costs of inventory systems. However, this inventory model assumes that all the leftover inventories are salvaged at the end of each period. This assumption is applicable to perishable products. For future research, extending this model to cases where leftover inventories to be carried to the next period is the future research direction.

Index Terms—Partial information, Markovian process, inventory planning and control

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I. INTRODUCTION

Inventory control is one of the most studied topics for the supply chain management. Both demand and supply information is fundamental for any inventory control decisions. Most works in the supply chain management literature assume that the supply capacity is either known to the buyer, or unlimited. Few of them consider the supply capacity information uncertainty and its influence to the inventory policy. Yet the supply capacity is always unknown or only partially observable in practice. In such an environment, most of well-known results may not hold. It is the objective of this paper to explore the structural properties of an inventory system with partially observed supply capacity and to study if there exists an optimal policy for the above problem.

A. The Importance of Modeling the Partially Observed Capacity

The study of inventory systems with the partially observed supply capacity is important in many real-life cases. We shall introduce some cases in a supply chain, where capacities can be partially observed.

Supplier’s capacity uncertainty: Unexpected machine breakdowns, resulting unexpected maintenance, may lead to a lower production capacity. Moreover, the uncertain repairing time may affect the availability of certain facilities: the repair is not planned, even planned, then its repairing time is uncertain. These stochastic events lead to an uncertain supply capacity. Further, the product quality can also be uncertain. When the product quality is low or the rate of defects is high, the actual available products are unknown. Since these uncertainties can’t be predicted, the supply capacity is not observed directly.

Buyers’ competition: A supplier serves multiple customers. When the capacity is tight, the supplier reserves its capacity to individual customer. It is known in practice as the capacity allocation, or rationing. Orders from buyers vary from time and are unknown to the other buyers. Hence the reserved capacity for an individual buyer is uncertain, even under condition of knowing the supply allocation rules.
As a result, a buyer doesn’t know the capacity volume reserved for him.

**Multiple sources of supply:** The buyer also orders from multiple suppliers. Each supplier has its own reserved capacity for the buyer. Hence the buyer can only estimate the capacity reserved for him, based on observed signals, such as the received ordering quantity. As a result, the available capacity is partially observed.

### B. Summary of Paper

This paper studies the optimal inventory replenishment policy at the beginning of each period to meet the stochastic demand with the partially observed supply capacity information. We allow the buyer to fulfill the out-of-capacity orders by paying an extra cost from other sources, such as a spot market. The demand is realized at the end of each period. The leftover inventory is salvaged while the unfulfilled demand incurs a penalty cost.

We consider a multi-period problem where the capacity observation of the current period will influence the capacity distribution of the next period, as well as the value function of the next period. The capacity allocated by the supplier may change from period to period, evolving as a Markovian Process. The capacity for the current period is NOT fully observed by the buyer at the time when the buyer places its orders. When the order quantity is greater than the supply capacity, the supplier provides its maximum capacity to the buyer. Only at this time, the capacity is fully observed by the buyer. The partially observed capacity information limits buyer’s capability in forecasting its needs as well as the inventory optimization.

In this paper, we prove the existence of an optimal ordering policy for a general Markovian capacity process. With results for the general case, we specialize our study to cases where the supply capacity has two possible states. We find out that revealing the supply capacity leads to a cost reduction. Hence the buyer may purchase more than needed in order to figure out the supply capacity. Moreover, if the possible capacity is low, the buyer’s ordering policy is equivalent to a myopic inventory ordering policy. Otherwise, the buyer makes the tradeoff between the current and the future period costs.

We also study the performance of the partial-observation model and the non-update model. In the partial-observation model, the buyer dynamically updates the supply capacity forecast according to the partial supply information based on the order fulfillment in each period. On the contrary, for non-update model, this partially observed supply capacity information is ignored. We prove the existence of the optimal inventory replenishment for the partial-observation model. And we use numerical experiments to demonstrate significant cost savings of this partial-observation model. **Hence, our finding suggests inventory managers pay a particular attention to the partially observed supply capacity information when there exists a supply capacity uncertainty.**

### C. Literature Review and Its Relation to Our Model

There is an extensive literature in inventory decisions with the given capacity constraint, such as the work of Federgruen and Zipkin [5], Gallego and Scheller-Wolf [8], Gallego and Toktay [9], and Gallien and Wein [10]. In addition, there are papers dealing with the uncertain capacity constraint. Khang and Fujiwara [7] address a discrete time inventory model where the maximum amount of supply, from which instantaneous replenishment orders can be made, is a random variable. They also characterize the optimal ordering policies. Wang and Gerchak [13] extends the model to include the variable capacity and random yield together in periodic inventory systems.

The research on inventory decisions with the partially observed information is quite recent. Bensoussan, Cakanyildirim and Sethi [4] study a multi-period news-vendor model with a partially observed demand. By assuming that the leftover inventories are salvaged and unsatisfied demands are lost in each period, they decouple the periods from the view point of the Bayesian demand update, and prove the existence of optimal ordering policy. Aviv and Pazgal [1] introduce the partially observed Markovian decision process. In their setting, to maximize its revenue, a buyer faces a dynamic pricing problem of selling a given stock during a finite selling season. Bensoussan, Cakanyildirim and Sethi [2] and [3] study the inventory system with partially observed inventory levels. Lu, Song and Zhu [11] and [12] study the news-vendor model with censored demand data.

To the best of our knowledge, the problem of inventory decisions with a partially observed Markovian capacity remains open. We model an inventory system with the partially observed Markov supply capacity. Similar to Bensoussan, Cakanyildirim and Sethi [4], the *unnormalized probability* is used to linearized the highly nonlinear dynamic programming equation. We solve the inventory management problem with the partial information of supply capacity, while Bensoussan, Cakanyildirim and Sethi [4] solve the inventory management problem with the partial information of demand. The methodology in Bensoussan, Cakanyildirim and Sethi [4] is extended in our model to prove the existence of optimal inventory policy. Moreover, an extensive model with two possible supply capacities is studied and the buyer’s tradeoff between the current and future period cost is characterized.

### D. Plan of Paper

In the next section, we formulate the problem. In our model, the evolution of partially observed Markovian ca-
capacity is characterized, and the un-normalized probability is used to linearize the dynamic equation. In Section 3, we prove the existence of an optimal ordering policy. In Section 4, we specialized the optimal inventory replenishment policy for the problem with two-state Markovian capacity. We also carry out numerical experiments. We find that ordering up to the smaller possible capacity can save the future cost. Finally, we conclude the paper in Section 5.

II. PROBLEM FORMULATION AND DYNAMIC PROGRAMMING EQUATIONS

A. Sequence of Events and Cost Equations

The sequence of events is as follows: at the beginning of each period, the buyer informs the supplier for its intended order quantity \( q_t \); thus the supplier checks the available capacity reserved for this buyer and tries its best to satisfy the requested purchase. Denote the reserved supply capacity as \( Q_t \). Note that \( \{ Q_t \} \) is a Markovian Process with a transition probability \( p(Q_{t+1} \mid Q_t) \), which is exogenously decided as an input. The buyer gets 1) \( a_t = q_t \wedge Q_t \) at the normal unit purchasing cost \( c \) and 2) \( q_t - a_t \), if the unsatisfied order is greater than 0, at a higher unit purchasing cost \( c + e \), where \( e \geq 0 \) indicates that the extra order can be more expensive. Note that the extra order may come from the same supplier by asking for an expensive, emergent order, or come from some other supply resources, such as a spot market. At the end of this selling season, all the unsatisfied demand is lost. Shortage cost at the unit cost of \( b \), and salvaged value at the unit revenue of \( h \) are recorded.

The following is the list of notations used in this paper:

- \( q_t \) : order quantity from the buyer;
- \( Q_t \) : available capacity of the supplier;
- \( \alpha \) : discount factor;
- \( D_t \) : demand of period \( t \);
- \( p(Q_{t+1} \mid Q_t) \) : transition probability of a Markov process \( \{ Q_t \} \);
- \( Pr\{ A \} \) : probability of the event \( A \);
- \( c \) : normal unit purchasing cost;
- \( a_t \) : amount of order charged at normal purchasing cost \( c \);
- \( e \) : additional unit purchasing cost for the extra unit when \( q_t > Q_t \);
- \( h \) : unit salvage value;
- \( b \) : unit shortage cost;
- \( F(\cdot) \) : cumulative demand distribution function;
- \( \pi_t(\cdot) \) : probability density function of capacity \( Q_t \);
- \( V_t(\cdot) \) : value function.

B. Dynamic forecast updates for the supply capacity

In each period, the buyer observes \( a_t \), which is the amount of ordering charged at the normal purchasing cost \( c \). Then, two scenarios are as follows.

1) When the order quantity is greater than the supply capacity, i.e. \( q_t \geq Q_t \), the supplier can’t satisfy the request. Then the supplier exhausts all of its capacity to produce products, i.e. \( a_t = Q_t \). For this scenario, the supply capacity reveals through the observed \( a_t \).

The density probability function of the capacity for the next period can be obtained as follows:

\[
\pi_{t+1}(Q_{t+1}) = p(Q_{t+1} \mid a_t); \quad (1)
\]

2) On the other hand, the order quantity is less than the supply capacity, and the supplier satisfies the request completely, i.e. \( q_t \leq Q_t \) and \( a_t = q_t \). For this scenario, the buyer knows that the capacity is greater than its order quantity, i.e. \( Q_t \geq a_t \), but doesn’t know the exact capacity. We denote this case as the partially observed supply capacity. Then

\[
\pi_{t+1}(Q_{t+1}) = \frac{Pr\{ Q_{t+1} | Q_t \geq a_t \} \cdot \pi_t(\xi) p(Q_{t+1} | \xi) d\xi}{\int_{a_t}^{\infty} \pi_t(\xi) d\xi}, \quad (2)
\]

where the second equality is due to Bayesian equations, and the third equality is due to the conditional probability equations, which is \( Pr\{ Q_{t+1} | Q_t \geq a_t \} = \int_{a_t}^{\infty} Pr\{ Q_{t+1} | Q_t = \xi \} \pi_t(\xi) d\xi \).

Hence the supply capacity \( Q_t \) is partially observed. This approach is similar to Bensoussan, Cakanyildirim and Sethi [4] where the partially observed demand signal is characterized.

C. Dynamic Programming Equations

With the sequence of events described above, it is possible for us to derive one-period cost function as \( L(D_t, q_t) + e(q_t - Q_t)^+ \), where

\[
L(x, y) = \begin{cases} 
 cy - h(y - x), & \text{if } x \leq y \\
 cy + b(x - y), & \text{if } y \leq x,
\end{cases} \quad (3)
\]

To eliminate trivial cases, it is reasonable to assume that \( 0 \leq h < c < b \) and \( 0 \leq e \). Then it is straightforward to see that

\[
L(x, y) \leq \begin{cases} 
 cy, & \text{if } x \leq y \\
 bx, & \text{if } y \leq x.
\end{cases} \quad (4)
\]
Recall that $\pi_t(\cdot)$ is the probability density function of the capacity $Q_t$. Then the value function can be expressed as follows:

$$V_t(\pi_t(x)) = \inf_{q_t \geq 0} \left\{ E_{D_t}[L(D_t, q_t)] + e(q_t - Q_t) + \alpha V_{t+1}(p(x | Q_t))\pi_t(Q_t)dQ_t + \int_{q_t}^{+\infty} E_{D_t}[L(D_t, q_t)] + \alpha V_{t+1}\left(\frac{\int_{q_t}^{+\infty} \pi_t(\xi)p(x | \xi)\xi d\xi}{\int_{q_t}^{+\infty} \pi_t(\xi)\xi d\xi}\right)\pi_t(Q_t)dQ_t \right\},$$

where the first term on the right hand side is the cost when the supply capacity $Q_t$ is smaller than the buyer’s intended ordering quantity, and the second term of $+ \int_{q_t}^{+\infty} E_{D_t}[L(D_t, q_t)] + \alpha V_{t+1}\left(\frac{\int_{q_t}^{+\infty} \pi_t(\xi)p(x | \xi)\xi d\xi}{\int_{q_t}^{+\infty} \pi_t(\xi)\xi d\xi}\right)\pi_t(Q_t)dQ_t$ is the cost when the supply capacity $Q_t$ is larger than the buyer’s intended ordering quantity.

The buyer chooses a set of purchasing quantities $\{q_t\}$ to minimize its total cost. Ordering too much results in a high purchasing cost; on the other hand, ordering too less results in a high penalty cost and failures to observe the supply capacity.

**D. Unnormalized Probability**

It is difficulty to directly solve the dynamic equation (5) owning to complex fractional parts. Both denominator and numerator include the decision variable $q_t$. In this part, we use the similar technique of a variable substitution as in Benoussan, Cakanyildirim and Sethi [4] so that the equation (5) can be expressed in a simpler and solvable form.

From equations (1) and (2), we know

$$\pi_{t+1}(x) = I_{a_t=1} \int_{q_t}^{+\infty} \pi_t(\xi)p(x | \xi)d\xi + I_{a_t<1} \int_{q_t}^{+\infty} \pi_t(\xi)d\xi,$$

where the indicator function $I_A = 1$ when event $A$ occurs, otherwise $I_A = 0$. Define recursively that

$$\rho_{t+1}(x) := I_{a_t=q_t} \int_{q_t}^{+\infty} p(x | \xi)\rho_t(\xi)d\xi \quad + I_{a_t<q_t}p(x | a_t), \quad t \geq 1;$$

$$\pi_1(x) := \pi_1(x).$$

Also let

$$\lambda_t := \int \rho_t(x)dx.$$ 

Then

$$\lambda_{t+1} = I_{a_t=q_t} \int_{q_t}^{+\infty} \rho_t(\xi)dx \quad + I_{a_t<q_t} \int \rho_t(x)dx.$$ 

We can check recursively (it holds when $t = 1$; and by supposing the correctness for $t$, check that it holds for $t+1$) that the following equation is true:

$$\rho_{t}(x) = \pi_t(x)\lambda_t.$$ 

Hence, we have

$$\pi_{t}(x) = \frac{\rho_{t}(x)}{\lambda_t} = \frac{\rho_{t}(x)}{\int \rho_{t}(x)dx}.$$ 

In what follows, we consider the infinite horizon case. To simplify the notation, the sub-script is omitted. We define $W(\rho)$ as

$$W(\rho) := \int \rho(x)dx \cdot V\left(\int \frac{\rho(x)}{\int \rho(x)dx}\right).$$

It is obvious that $W(\cdot)$ can be obtained by $V(\cdot)$. By definition of $W(\rho)$, we have

$$W(\rho) = \int \rho(x)dx \cdot \inf_{q \geq 0} \left\{ L(D, q)f(D)dD + \int_{0}^{q} e(q - Q)\rho(Q)dQ + \alpha \int_{0}^{q} V(p(\cdot | Q))\rho(Q)dQ + \alpha V\left(\int_{q}^{+\infty} \rho(\xi)p(\cdot | \xi)d\xi\right)\int_{q}^{+\infty} \rho(Q)dQ \right\},$$

$$= \inf_{q \geq 0} \left\{ \int L(D, q)f(D)dD \int \rho(x)dx + \int_{0}^{q} e(q - Q)\rho(Q)dQ + \alpha \int_{0}^{q} W(p(\cdot | Q))\rho(Q)dQ + \alpha W\left(\int_{q}^{+\infty} \rho(\xi)p(\cdot | \xi)d\xi\right) \right\}.$$

Note that Equation (7) is a Bellman equation in $\rho$. 


III. THE EXISTENCE OF AN OPTIMAL FEEDBACK SOLUTION

In this section, we first show the existence and uniqueness of the solution \( W \) for equation (7). Then we prove the existence of an optimal feedback control \( q^* \).

A. Existence of a Unique Solution \( W \)

In this subsection we start with a definition of a function space \( \mathcal{B} \). Then we prove that, if there is a solution to equation (7), it must exist in the above defined function space. As a result, the existence and the uniqueness of the solution follow. Note that similar definitions of function spaces \( \mathcal{H} \) and \( \mathcal{B} \) are used in Bensoussan, Cakanyildirim and Sethi [4].

Define function space \( \mathcal{H} \) of function \( \rho \):

\[
\mathcal{H} := \{ \rho \in L^1(\mathbb{R}^+) : \int_0^\infty x|\rho(x)|dx < \infty \},
\]

(8)

where \( L^1(\mathbb{R}^+) \) is the space of integrable functions whose domain is the set of nonnegative real numbers, and

\[
\mathcal{H}^+ := \{ \rho \in \mathcal{H} | \rho \geq 0 \},
\]

(9)

with the norm:

\[
||\rho|| = \int_0^{+\infty} |\rho(x)|dx + \int_0^\infty x|\rho(x)|dx.
\]

(10)

Also define the following space \( \mathcal{B} \) of function \( \phi \):

\[
\mathcal{B} = \left\{ \phi(\rho) : \mathcal{H}^+ \to \mathcal{R} \left| \sup_{x>0} |\phi(\rho)| < \infty \right. \right\},
\]

(11)

with the norm:

\[
||\phi||_\mathcal{B} = \sup_{\rho \in \mathcal{H}^+} \frac{|\phi(\rho)|}{||\rho||},
\]

(12)

For the technical convenience, we need the following assumptions:

**Assumption 3.1**: Assume, for any \( \rho \in \mathcal{H}^+ \),

\[
\int x \int p(x|\xi)\rho(\xi)d\xi dx \leq c_0 \int \xi \rho(\xi)d\xi,
\]

(13)

with \( c_0 < 1 \).

This assumption is necessary to complete proofs of the following lemmas and the theorem. It is satisfied by specific probability transition function \( p(\cdot|\cdot) \), such that

\[
\alpha \int x \int p(x|\xi)\rho(\xi)d\xi dx \leq c_0 \int \xi \rho(\xi)d\xi < \int \xi \rho(\xi)d\xi,
\]

i.e.

The discounted expected demand of next period forecasted by \( p(\cdot|\cdot) \) is equal to the mean of current period demand.

Since the discount factor is always strictly smaller than 1, this assumption holds when the expected demand of next period forecasted by \( p(\cdot|\cdot) \) is equal to the mean of current period demand.

We need the following lemma. Its proof appears in Appendix.

**Lemma 3.1**: If equation (7) has a solution \( W \), the solution is in \( \mathcal{B} \).

Now define the map \( T(W) \) as,

\[
T(W) := \min_{q \geq 0} \left\{ \int L(D,q)f(D)dD \int \rho(x)dx + \int_0^q e(q-Q)\rho(Q)dQ + \alpha \int_0^q W(p(\cdot|Q))\rho(Q)dQ + \alpha W(\int_q^{+\infty} \rho(\xi)p(\cdot|\xi)d\xi) \right\}.
\]

(15)

Then we can obtain the following lemma. See Appendix for its proof.

**Lemma 3.2**: \( ||T(W) - T(\bar{W})||_\mathcal{B} \leq \alpha \max\{1,c_0\}||W - \bar{W}||_\mathcal{B} \).

**Theorem 3.1**: There exists a unique solution \( W \) of the dynamic programming equation (7).

**Proof**: From Assumption 3.1, we know that \( \alpha c_0 < 1 \), such that \( \alpha \max\{1,c_0\} < 1 \). Hence by Lemma 3.2, the mapping \( T : \mathcal{B} \to \mathcal{B} \) is a contraction mapping. Then by the Contraction Mapping Theorem [6], there exists a unique solution \( W \) such that \( T(W) = W \). Hence we have proved the desired results.

B. The existence of Optimal Feedback Control

By Equation (3), we can easily obtain \( L(x,y) \geq (c-h)y \). Substitute it into Equation (7), we then can prove that \( W(\rho) \geq (c-h)q + \rho(x)dx \). With equations (52) and (56), we know that

\[
(c-h)q \int \rho(x)dx \leq W(\rho)
\]

\[
\leq \left( \frac{b\mu_D}{\mu_Q} + \alpha \max\{1,c_0\}||W||_\mathcal{B} \right) ||\rho||
\]

\[
\leq \left( \frac{b\mu_D}{\mu_Q} + \alpha \max\{1,c_0\} \frac{b\mu_D}{\mu_Q(1-\alpha \max\{1,c_0\})} \right) ||\rho||.
\]

Note that \( \int \rho(x)dx = \frac{\int x\rho(x)dx}{\int \rho(x)dx} = \frac{\int x\rho(x)dx}{\mu_Q} = 1 + \mu_Q \). Hence we obtain that

\[
0 \leq q \leq \frac{b\mu_D(1 + \mu_Q)}{\mu_Q(c-h)(1-\alpha \max\{1,c_0\})} := B.
\]

(16)
Now we define $W_B(\rho)$ as
\[
W_B(\rho) = \min_{q \in [0,B]} \left\{ \int L(D,q) f(D) dD \int \rho(x) dx + \int_0^q e(q - Q) \rho(Q) dQ + \alpha \int_0^q W_B(\rho(\cdot | Q)) \rho(Q) dQ + \alpha W_B(\int_0^{+\infty} \rho(\xi) p(\cdot \mid \xi) d\xi) \right\}.
\]  
(17)

Note that $W_B(\rho)$ can be shown to be a unique solution to equation (17) since $W(\rho)$ is shown to be unique by Lemma 3.2. If $W_B(\rho)$ is continuous in $\rho$, we obtain the continuity of $G(q)$ (defined in equation (7)) in $q$. After that, the existence of the minimizer $q^*$ is established. See Appendix for proofs of the following lemma.

**Lemma 3.3:** For any $\rho, \tilde{\rho} \in \mathcal{H}^+$, we have
\[
|W_B(\rho) - W_B(\tilde{\rho})| \leq H_B ||\rho - \tilde{\rho}||,
\]  
(18)

where $H_B$ is a constant which is independent of $\rho$.

**Theorem 3.2:** There exists an optimal feedback control, i.e., the optimal order quantity.

**Proof:** Recall the definition of $G(q)$ in Equation (7), and we can write
\[
W_B(\rho) = \min_{q \in [0,B]} G(q),
\]  
(19)

where
\[
G(q) = \int L(D,q) f(D) dD \int \rho(x) dx + \int_0^q e(q - Q) \rho(Q) dQ + \alpha \int_0^q W_B(\rho(\cdot | Q)) \rho(Q) dQ + \alpha W_B(\int_0^{+\infty} \rho(\xi) p(\cdot \mid \xi) d\xi).
\]  
(20)

The first, second, and third terms of equation (20) are continuous in $q$. By Lemma 3.3, we know that $W(\rho)$ is continuous in $\rho$, which leads to the continuity of the fourth term of equation (20) in $q$. Hence $G(q)$ is continuous in $q$. Moreover, $q$ belongs to a bounded and closed set $[0,B]$. Hence, by Weierstrass’ Theorem [6], there exists a minimum in the bounded and closed set to minimize such a continuous function. The minimum $q^*$ can be obtained.

**IV. Optimal Policy Analysis of Two-State Markov Chain**

In order to further analyze the optimal purchasing policy, and reveal the supply capacity, we specialize our study to cases where the supply capacity can only take two possible values of $Q_t$ and $Q_h$ with $Q_t < Q_h$. Note that three or more finite state problem can be analyzed by the similar approach, with much more tedious forms and equations. We assume that the un-normalized capacities are $Q_t$ and $Q_h$, with probability of $\theta_t$ and $\theta_h$, respectively. Hence the distribution function can be written as
\[
\rho(x) = \theta_t I_{x = Q_t} + \theta_h I_{x = Q_h}.
\]  
(21)

Suppose that the transition probability functions are
\[
p(x|Q_t) = \beta_t I_{x = Q_t} + (1 - \beta_t) I_{x = Q_h},
p(x|Q_h) = (1 - \beta_h) I_{x = Q_t} + \beta_h I_{x = Q_h},
\]  
(22)

that is to say, the transition matrix is
\[
B := \begin{pmatrix} \beta_t & 1 - \beta_t \\ 1 - \beta_h & \beta_h \end{pmatrix}.
\]

Then the dynamic programming Equation (7) can be further reduced to
\[
W(\rho) = \min_{q \geq 0} \left\{ [cq - h] \int_0^q (q - D) f(D) dD + b \int_q^{+\infty} (D - q) f(D) dD (\theta_t +\theta_h) + \int_{Q_t \leq q < Q_h} e(q - Q) \theta_t \right. \\
+ \int_{Q_h \leq q < \infty} e(q - Q)(\theta_t + e(q - Q_h)\theta_h) + \alpha \int_{Q_t \leq q < Q_h} W(\rho(\cdot | Q_t)) + e q_{Q_h < \infty} W(\rho(\cdot | Q_h)) \left. \right\}.
\]  
(23)

With the definition of $W(\rho)$ in Equation (6), we obtain
\[
W(\alpha \rho(\cdot)) = a W(\rho(\cdot)), \quad \text{if } a \text{ is a constant.}
\]  
(24)

By collecting and simplifying terms, we have
\[
W(\rho) = \min_{q \geq 0} \left\{ [cq - h] \int_0^q (q - D) f(D) dD + b \int_q^{+\infty} (D - q) f(D) dD (\theta_t +\theta_h) + \int_{Q_t \leq q < Q_h} e(q - Q) \theta_t \right. \\
+ \int_{Q_h \leq q < \infty} e(q - Q)(\theta_t + e(q - Q_h)\theta_h) + \alpha \int_{Q_t \leq q < Q_h} W(\rho(\cdot | Q_t)) + e q_{Q_h < \infty} W(\rho(\cdot | Q_h)) \left. \right\}.
\]  
(25)

Note that the aim of observing current period supply capacity is to predict supply capacity for future periods. And with this prediction, it is possible for us to adjust our purchasing quantity to minimize the cost function. In order to characterize the purchasing policy which influences the costs for the current (myopic cost) and the future period.
(cost-to-go from the next period), we divide the total cost function into two parts: \( P_c(q) \) and \( P_f(q) \). Define
\[
P_c(q) := [cq - h \int_0^q (q - D) f(D) dD + b \int_q^{+\infty} (D - q) f(D) dD] (\theta_l + \theta_h) + I_{Q_l \leq q < Q_h} e(q - Q_l) \theta_l + I_{Q_h \leq q} e(q - Q_h) \theta_h,
\]
and
\[
P_f(q) := \alpha I_{Q_l < q} W(p(\cdot|Q_l)) \theta_l + W(p(\cdot|Q_h)) \theta_h + \alpha I_{Q_l \leq q} W[\theta_l p(\cdot|Q_l) + \theta_h p(\cdot|Q_h)].
\]
Then the total cost can be expressed as follows:
\[
W(\rho) = \min_{q \geq 0} \{P_c(q) + P_f(q)\}. \quad (26)
\]

A. The case of the Independent Capacity

When the prevailing future capacity is independent to the current capacity, we denote this case as the case of the independent capacity. Mathematically, the transition probabilities are chosen as \( \beta_l = 1 - \beta_h \), which leads to \( p(\cdot|Q_l) = p(\cdot|Q_h) \). With Expression (24), \( W[\theta_l p(\cdot|Q_l) + \theta_h p(\cdot|Q_h)] = (\theta_l + \theta_h) W(p(\cdot|Q_l)) \). Hence
\[
P_f(q) = \alpha I_{Q_l < q} W(p(\cdot|Q_l)) \theta_l + W(p(\cdot|Q_h)) \theta_h + \alpha I_{Q_l \leq q} W[\theta_l p(\cdot|Q_l) + \theta_h p(\cdot|Q_h)]
= \alpha (\theta_l + \theta_h) W(p(\cdot|Q_l)), \quad (27)
\]
which is independent of \( q \). No matter how the purchasing policy is, the value function doesn’t change at all. Based on this, \( W(\rho) \) can be reduced as follows:
\[
W(\rho) = \min_{q \geq 0} \{P_c(q)\} + P_f. \quad (28)
\]

Hence the myopic solution is optimal for the case of the independent capacity. In other words, the buyer only needs to focus on reducing the current period’s cost. The resulting myopic policy is also optimal in the view of multi-period. This result is intuitively right. In what follows, we figure out what exactly \( q^* \) is, and how other factors influence the optimal purchasing quantity.

Define
\[
g_1(q) := cq - h \int_0^q (q - D) f(D) dD + b \int_q^{+\infty} (D - q) f(D) dD (\theta_l + \theta_h);
\]
its convexity in \( q \) is straightforward, and
\[
g_2(q) := I_{Q_l \leq q < Q_h} e(q - Q_l) \theta_l + I_{Q_h \leq q} e(q - Q_h) \theta_h.
\]
Then \( P_c(q) = g_1(q) + g_2(q) \). The minimizer of function \( g_1(q) \) is
\[
F^{-1} \left( \frac{b - c}{b - h} \right).
\]
g_2(q) is a piece-wise linear function with minimizers less than or equal to \( Q_l \).

For the notation convenience, we further define
\[
\tilde{q}_c := F^{-1} \left( \frac{b - c}{b - h} \right),
\]
\[
\tilde{q}_{c1} := F^{-1} \left( \frac{b - c - e^{-\theta_l}}{b - h} \right),
\]
\[
\tilde{q}_{c2} := F^{-1} \left( \frac{b - c - e^{-\theta_h}}{b - h} \right). \quad (29)
\]
Obviously, we have \( \tilde{q}_c > \tilde{q}_{c1} > \tilde{q}_{c2} \).

1) Case 1: \( \tilde{q}_c < Q_l \): Since \( \tilde{q}_c \) minimizes both \( g_1(q) \) and \( g_2(q) \), the minimizer
\[
q^* = \tilde{q}_c = F^{-1} \left( \frac{b - c}{b - h} \right). \quad (30)
\]

2) Case 2: \( Q_l \leq \tilde{q}_c < Q_h \): In this case, since \( g_1(q) \) and \( g_2(q) \) are non-increasing for \( q \in [0, Q_l] \) and non-decreasing for \( q \in [Q_h, +\infty) \), we can narrow down our search space to \([Q_l, Q_h]\). That is
\[
\min_{q \in [Q_l, Q_h]} \{ [cq - h \int_0^q (q - D) f(D) dD + b \int_q^{+\infty} (D - q) f(D) dD (\theta_l + \theta_h) + e(q - Q_l) \theta_l + e(q - Q_h) \theta_h] \}. \quad (31)
\]
Let us solve the first order condition, and we have
\[
F^{-1} \left( \frac{b - c - e^{-\theta_l}}{b - h} \right) = \tilde{q}_{c1} < \tilde{q}_c < Q_h.
\]
Hence the optimality is obtained at the point \( \tilde{q}_{c1} \) or the lower bound of \( Q_l \), i.e.
\[
q^* = \max \{ \tilde{q}_{c1}, Q_l \} = \max \{ F^{-1} \left( \frac{b - c - e^{-\theta_l}}{b - h} \right), Q_l \}. \quad (32)
\]

3) Case 3: \( Q_h \leq \tilde{q}_c \): In this case, since \( g_1(q) \) and \( g_2(q) \) are non-increasing for \( q \in [0, Q_l] \), similarly, we narrow down our search space to \([Q_l, +\infty) \). Note that \( g_1(q) + g_2(q) \) is equal to equation (31) for \( Q \in [Q_l, Q_h] \).

For \( q \in [Q_h, +\infty) \), \( g_1(q) + g_2(q) \) is
\[
\min_{q \in [Q_h, +\infty]} \{ [cq - h \int_0^q (q - D) f(D) dD + b \int_q^{+\infty} (D - q) f(D) dD (\theta_l + \theta_h) + e(q - Q_l) \theta_l + e(q - Q_h) \theta_h] \}, \quad (33)
\]
which is convex in \( q \) and the solution of the first order condition is \( \tilde{q}_{c2} \) (defined in equation (29)). Hence the optimal purchasing in this case is as follows:
1) \( \bar{q}_{c2} > Q_h \): since \( \bar{q}_{c2} > Q_h \), we have \( \bar{q}_{c1} > Q_h \), which indicates that \( g_1(q) + g_2(q) \) is non-increasing for \( q \in [Q_l, Q_h] \). Then the optimal purchasing quantity is order-up-to \( \bar{q}_{c2} \), i.e., \( q^* = \bar{q}_{c2} \).

2) \( \bar{q}_{c1} > Q_h \) and \( \bar{q}_{c2} \leq Q_h \): it indicates that \( g_1(q) + g_2(q) \) is non-decreasing for \( q \in [Q_h, +\infty) \). \( \bar{q}_{c1} > Q_h \) indicates that \( g_1(q) + g_2(q) \) is non-increasing for \( q \in [Q_l, Q_h] \). Then the optimal purchasing quantity is order-up-to \( Q_h \), i.e., \( q^* = Q_h \).

3) Otherwise \( \bar{q}_{c1} \leq Q_h \): this is similar to Case 2 and the optimal purchasing quantity is order-up-to \( \max\{\bar{q}_{c1}, Q_l\} \), i.e., \( q^* = \max\{\bar{q}_{c1}, Q_l\} \).

We conclude the optimal purchasing quantity as follows:

\[
q^* = \begin{cases} 
\max\{\bar{q}_{c2}, Q_h\}, & \text{if } \bar{q}_{c1} > Q_h; \\
\max\{\bar{q}_{c1}, Q_l\}, & \text{if } \bar{q}_{c1} \leq Q_h.
\end{cases}
\] (34)

To conclude this subsection, combining 3 cases together, we have the following theorem:

**Theorem 4.1:** The optimal purchasing quantity \( q^* \) for the independent capacity case is as follows,

<table>
<thead>
<tr>
<th>Scenarios</th>
<th>Conditions</th>
<th>Optimal purchasing ( q^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>( \bar{q}<em>c &lt; Q_l ) and ( \bar{q}</em>{c1} = \bar{q}_{c2} )</td>
<td>( \bar{q}_c )</td>
</tr>
<tr>
<td>(2)</td>
<td>( Q_h \geq \bar{q}<em>{c1} ) and ( \bar{q}</em>{c2} &gt; Q_l )</td>
<td>( \max{\bar{q}_{c1}, Q_l} )</td>
</tr>
<tr>
<td>(3)</td>
<td>( Q_h )</td>
<td>( \max{\bar{q}_{c2}, Q_h} )</td>
</tr>
</tbody>
</table>

where \( \bar{q}_c, \bar{q}_{c1}, \bar{q}_{c2} \) are defined by equation (29) and \( Q_l, Q_h \) are two possible supply capacities with \( Q_h > Q_l \). See Figure 1 for an illustration of the possible scenarios.

Comparing with possible supply capacities, when the demand is low (\( \bar{q}_c < Q_l \)) and \( \bar{q}_{c1} < Q_h \), we simply purchase up to the minimizer \( \bar{q}_c \) at the normal price without considering the capacity constraint.

When the demand is moderate, i.e., the minimizer \( \bar{q}_c \) of the normal price is greater than the smaller possible capacity \( Q_l \), and the minimizer \( \bar{q}_{c1} \) of the high price is smaller than the larger possible capacity \( Q_h \), we order up to the larger one between \( \bar{q}_{c1} \) and \( Q_l \).

When the demand is forecasted to be high, i.e., the cost minimizer \( \bar{q}_c \) of the normal price is greater than the smaller possible capacity \( Q_l \), and the cost minimizer \( \bar{q}_{c1} \) of the high price is as well greater than the bigger possible capacity \( Q_h \), we order up to the larger one between \( \bar{q}_{c2} \) and \( Q_h \).

Moreover, when \( \theta_l = 0 \), i.e., the only possible capacity is \( Q_h \), then it is easy to obtain from Theorem 4.1 that

\[
q^* = \begin{cases} 
\bar{q}_c, & \text{if } \bar{q}_c \leq Q_h; \\
\max\{\bar{q}_{c2}, Q_h\}, & \text{otherwise}.
\end{cases}
\]

Further, if the additional unit purchasing cost \( e = +\infty \), which indicates that it is impossible to make the additional order, the optimal policy is

\[
q^* = \min\{\bar{q}_c, Q_h\}.
\]

Theorem 4.1 characterizes a myopic policy and establishes a foundation for the subsequent discussions.

**B. General Case: Purchase More to Observe Supply Capacity**

In this subsection, we consider the cases where the supply capacity of the next period is correlated to the capacity of the current period. If capacities are correlated, then the future cost function \( P_f \) has to be taken into consideration. First, we need the following lemma.

**Lemma 4.1:** The solution \( W(\rho) \) of equation (7) is concave in \( \rho \), i.e.,

\[
W(\beta \rho + (1 - \beta) \hat{\rho}) \geq \beta W(\rho) + (1 - \beta) W(\hat{\rho}).
\]

**Proof:** Note that the uniqueness of \( W(\rho) \) is proved in Theorem 3.1. We use the mapping \( T \) defined in equation (15) and prove the lemma by value iteration according to \( W^{n+1} = TW^n \). Let \( W^0(\rho) = 0 \), which is obviously concave in \( \rho \). Suppose \( W^n(\rho) \) is concave in \( \rho \), then

\[
W^{n+1}(\beta \rho + (1 - \beta) \hat{\rho}) = \min_{\rho \geq 0} \left\{ \int_L L(D, \rho) f(D) dD \int (\beta \rho(x) + (1 - \beta) \hat{\rho}(x)) dx + \int_0^q e(q - Q) (\beta \rho(Q) + (1 - \beta) \hat{\rho}(Q)) dQ + \alpha \int_0^q W^n(p(\cdot \mid Q)) (\beta \rho(Q) + (1 - \beta) \hat{\rho}(Q)) dQ + \alpha W^n(\int_{q}^{\infty} (\beta \rho(\xi) + (1 - \beta) \hat{\rho}(\xi)) p(\cdot \mid \xi) d\xi) \right\}
\]

\[
\geq \min\{\beta T_{\rho}(W^n(\rho)) + (1 - \beta) T_{\hat{\rho}}(W^n(\rho))\}
\]

Since \( W^n(\rho) \) converges to the solution \( W(\rho) \), we have the desired result.

Then based on this fact, define

\[
K_l := \alpha W(\theta_l p(\cdot \mid Q_l) + \theta_h p(\cdot \mid Q_h)) - \alpha [\theta_l W(p(\cdot \mid Q_l)) + \theta_h W(p(\cdot \mid Q_h))]
\]

where the inequality is due to Lemma 4.1.
Recall the definition of $P_f$, and it is straightforward to conclude the following theorem.

Theorem 4.2: $P_f(q_{m}) = P_f(q_{l}) - K_l$, for all $q_{m} \geq Q_l$, and $0 \leq q_{l} < Q_l$.

This theorem demonstrates that the buyer may try to purchase more to reduce the future cost at the amount of $K_l$. This is because purchasing more than $Q_l$ leads to the buyer to figure out whether the current period capacity is $Q_l$. If it isn’t $Q_l$, the only possible value of $Q$ is $Q_h$. Hence ordering up to $Q_l$ can fully discover the exact value of the current period’s capacity, which leads to a reduction of the cost-to-go function.

Since the cost-to-go function remains the same when $q \in [Q_l, +\infty)$, we know that the purchasing policy does not change in scenarios (2) and (3) described in Theorem 4.1. When scenario (1) occurs, we need to balance the tradeoff between myopic and cost-to-go function. Define the myopic cost reduction as

$$K_m := g_1(Q_l) + g_2(Q_l) - g_1(\tilde{q}_c) + g_2(\tilde{q}_c) = g_1(Q_l) - g_1(\tilde{q}_c) > 0. \quad (37)$$

The buyer can reduce this myopic cost by ordering up to the cost minimizer $\tilde{q}_c$ instead of the smaller possible supply capacity $Q_l$.

Remark 4.1: The larger the gap between $Q_l$ and $\tilde{q}_c$ is, the greater the myopic cost reduction $K_m$ is.

Remark 4.2: The greater the second derivative of the cost function $g_1(\cdot)$, i.e., $\frac{d^2 g_1(q)}{dq^2}$, the greater the myopic cost saving $K_m$ is.

Then the optimal ordering quantity is

$$q^* = \begin{cases} \tilde{q}_c, & \text{if } K_m > K_l; \\ Q_l, & \text{if } K_m \leq K_l. \end{cases} \quad (38)$$

Theorem 4.3: The optimal purchasing quantity $q^*$ of the multi-period news-vendor problem with partially observed two-state Markovian capacity is as follows.

<table>
<thead>
<tr>
<th>Scenarios</th>
<th>Conditions</th>
<th>Optimal purchasing $q^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>$Q_h \geq \tilde{q}_c$ and $Q_c &gt; Q_l$</td>
<td>$\tilde{q}_c$, if $K_m &gt; K_l$; $Q_l$, if $K_m \leq K_l$.</td>
</tr>
<tr>
<td>(2)</td>
<td>$Q_h &lt; \tilde{q}_c$</td>
<td>$\tilde{q}_c$, if $K_m &gt; K_l$; $Q_l$, if $K_m \leq K_l$.</td>
</tr>
<tr>
<td>(3)</td>
<td>$Q_h &lt; \tilde{q}_c$</td>
<td>$\tilde{q}_c$, if $K_m &gt; K_l$; $Q_l$, if $K_m \leq K_l$.</td>
</tr>
</tbody>
</table>

where $\tilde{q}_c$, $\tilde{q}_1$, $\tilde{q}_2$ are defined in equation (29), $K_l$ and $K_m$ are defined in equations (36) and (37), respectively. $Q_l$ and $Q_h$ are two possible supply capacities with $Q_h > Q_l$.

See Figure 1 for the illustration of the above three possible scenarios.

Remark 4.3: When $\theta_l = 0$, the only possible capacity is $Q_h$. Then $K_l = 0$ and the policy is the same as the one of the basic model, the case of the independent supply capacity. The myopic policy is also optimal for the multi-period model.

Remark 4.4: When the demand is low (Scenario (1)), in order to save the cost-to-go from the next period, the buyer purchases products only up to the smaller possible supply capacity $Q_l$.

Remark 4.5: When the demand is low (Scenario (1)), the larger the gap is between $\tilde{q}_c$ and $Q_l$, the higher the possible is to order up to $\tilde{q}_c$, which is the same as the myopic policy.

Remark 4.6: When the smaller possible supply capacity $Q_l$ is very low (Scenario (2) and (3)), the optimal policy is the same as the myopic one.

C. Numerical Analysis and Results

It is possible to re-write equation (25) in an explicit form in order to carry out numerical studies. By substituting Equation (21) and using Equation (24), we have

$$W(\theta_l I_{x=Q_l} + \theta_h I_{x=Q_h}) = \theta_l W(\frac{\theta_l I_{x=Q_l}}{\theta_l} + \sum I_{x=Q_h}) = \min_{q \geq 0} \left\{ \left[ eq - h \int_0^q (q - D) f(D) dD + b \int_{q}^{+\infty} (D - q) f(D) dD \right] \theta_l + \sum \right\}.$$

Further define

$$\theta = \frac{\theta_l}{\theta_h}, \quad \Phi(x) = W(x I_{x=Q_l} + I_{x=Q_h}), \quad (40)$$

and then we obtain

$$\Phi(\theta) = \min_{q \geq 0} \left\{ \left[ eq - h \int_0^q (q - D) f(D) dD + b \int_{q}^{+\infty} (D - q) f(D) dD \right] \theta_l + \sum \right\}.$$

Assumption 4.1: Assume a symmetric Markov chain of capacity: $\beta_l = \beta_h = \beta$. 

$$\Phi(\theta) = \min_{q \geq 0} \left\{ \left[ eq - h \int_0^q (q - D) f(D) dD + b \int_{q}^{+\infty} (D - q) f(D) dD \right] \theta_l + \sum \right\}.$$
Then we can rewrite (41) as:

\[ \Phi(\theta) = \min_{Q, \theta \geq 0} \left\{ [c-q-h \int_0^q (q-D) f(D) dD + b \int_q^{+\infty} (D-q) f(D) dD] (\theta + 1) + I_{q < Q_h} \theta + I_{Q_h \leq q < Q_l} (\theta + e(q - Q_l) + e(q - Q_h)) + \alpha \Phi(q) \Phi^n (\theta(1-\beta) + \beta (\frac{\beta}{1-\beta} + \Phi^n (\frac{1-\beta}{1-\beta}))) + \alpha (\theta(1-\beta) + \beta) I_{q \leq Q_l} \Phi^n (\frac{\theta + (1-\beta)}{\theta(1-\beta) + \beta}) \right\} \]  

(42)

Let \( \beta = \frac{3}{5}, h = 1, c = 2, b = 4, e = 1, Q_l = 12, Q_h = 15, \alpha = 0.95, \) demand \( D \) subjects to a normal distribution with mean \( \mu = 10 \) and variance \( \sigma = 1 \). Then we find that \( \bar{q} = 10.43 < Q_l \) (Here we only do analysis of this case because other cases’ policy is the same as the myopic inventory decision according to Remark 4.6).

We use value-iteration method to solve this equation. Start with \( \Phi^0(\theta) = 0 \) and

\[ \Phi^{n+1}(\theta) = \min_{q, \theta \geq 0} \left\{ [c-q-h \int_0^q (q-D) f(D) dD + b \int_q^{+\infty} (D-q) f(D) dD] (\theta + 1) + I_{q < Q_h} \theta + I_{Q_h \leq q < Q_l} (\theta + e(q - Q_l) + e(q - Q_h)) + \alpha I_{q < Q_h} \Phi^n (\frac{\theta(1-\beta) + \beta (\frac{1-\beta}{1-\beta} + \Phi^n (\frac{1-\beta}{1-\beta}))) + \alpha (\theta(1-\beta) + \beta) I_{q \leq Q_l} \Phi^n (\frac{\theta + (1-\beta)}{\theta(1-\beta) + \beta}) \right\} \]

(43)

In order to carry out the iteration process, let \( \theta = 2 \), then

\[ \Phi^{n+1}(2) = \min_{q, \theta \geq 0} \left\{ [c-q-h \int_0^q (q-D) f(D) dD + b \int_q^{+\infty} (D-q) f(D) dD] (\theta + 1) + I_{Q_h \leq q < Q_l} \theta + I_{Q_h \leq q < Q_l} (\theta + e(q - Q_l) + e(q - Q_h)) + I_{Q_h < q < Q_l} (\theta + e(q - Q_l) + e(q - Q_h)) + \alpha I_{Q_h < q < Q_l} \Phi^n (\frac{\theta(1-\beta) + \beta (\frac{1-\beta}{1-\beta} + \Phi^n (\frac{1-\beta}{1-\beta}))) + \alpha (\theta(1-\beta) + \beta) I_{q \leq Q_l} \Phi^n (\frac{\theta + (1-\beta)}{\theta(1-\beta) + \beta}) \right\} \]

(44)

And let \( \theta = \frac{1}{2} \), then

\[ \Phi^{n+1}(\frac{1}{2}) = \min_{q, \theta \geq 0} \left\{ [c-q-h \int_0^q (q-D) f(D) dD + b \int_q^{+\infty} (D-q) f(D) dD] (\theta + 1) + I_{Q_h \leq q < Q_l} \theta + I_{Q_h \leq q < Q_l} (\theta + e(q - Q_l) + e(q - Q_h)) + I_{Q_h < q < Q_l} (\theta + e(q - Q_l) + e(q - Q_h)) + \alpha I_{Q_h < q < Q_l} \Phi^n (\theta(1-\beta) + \beta (\frac{1-\beta}{1-\beta} + \Phi^n (\frac{1-\beta}{1-\beta}))) + \alpha (\theta(1-\beta) + \beta) I_{q \leq Q_l} \Phi^n (\frac{\theta + (1-\beta)}{\theta(1-\beta) + \beta}) \right\} \]

(45)

Also, let \( \theta = 1 \) and we have

\[ \Phi^{n+1}(1) = \min_{q, \theta \geq 0} \left\{ [c-q-h \int_0^q (q-D) f(D) dD + b \int_q^{+\infty} (D-q) f(D) dD] (\theta + 1) + I_{Q_h \leq q < Q_l} \theta + I_{Q_h \leq q < Q_l} (\theta + e(q - Q_l) + e(q - Q_h)) + I_{Q_h < q < Q_l} (\theta + e(q - Q_l) + e(q - Q_h)) + \alpha I_{Q_h < q < Q_l} \Phi^n (\theta(1-\beta) + \beta (\frac{1-\beta}{1-\beta} + \Phi^n (\frac{1-\beta}{1-\beta}))) + \alpha (\theta(1-\beta) + \beta) I_{q \leq Q_l} \Phi^n (\frac{\theta + (1-\beta)}{\theta(1-\beta) + \beta}) \right\} \]

(46)

Note that \( \Phi(\frac{1}{2}) \), \( \Phi(\frac{3}{5}) \) and \( \Phi(1) \) appear in equations (44), (45), and (46), respectively. In order to carry out the iteration, we make the following approximations:

\[ \Phi(1) \approx \Phi(\frac{4}{5}) \approx \Phi(\frac{1}{2}) \]

Then combining equations (44), (45), and (46) together, starting with \( \Phi^0(1) = 0, \Phi^0(2) = 0 \) and \( \Phi^0(\frac{1}{2}) = 0 \), we solve the equation numerically and obtain

\[ \Phi(\frac{1}{2}) = 416.8 < \Phi(1) = 547.6 < \Phi(2) = 756.8, \]  

(47)

where we find out that \( \Phi(1) \geq \frac{1}{2} \Phi(\frac{1}{2}) + \frac{1}{4} \Phi(2) \) (demonstrating concavity), and the optimal purchasing

\[ q^*|_{\theta = \frac{1}{2}} = 12, \quad q^*|_{\theta = 1} = 12, \quad q^*|_{\theta = 2} = 10.43. \]  

(48)

With the above calculation, the optimal purchasing is \( Q_l = 12 \) or \( \bar{q} = 10.43 \) when \( \bar{q} < Q_l \). And also the larger \( \theta = \frac{\alpha}{\sigma} \).
is, the larger the cost $\Phi$ is. This is intuitively true. A larger value of $\frac{\theta}{\theta_n}$ indicates a higher possibility that $Q_1$ is the next period’s capacity, which leads to a higher purchasing costs.

D. Comparison with the non-update model

Denote the model discussed before as the partial-observation model, since the buyer dynamically updates the supply capacity forecast after observing the partial supply capacity information. Now we compare it to a non-update model, where the buyer doesn’t consider the capacity observation issue when placing orders, and the next period capacity distribution is $\theta_l p(\cdot | Q_t) + \theta_h p(\cdot | Q_h)$. Then Equation (25) can be rewritten as

$$
\bar{W}(\rho) = \min_{q \geq 0} \left\{ [cq - h \int_0^q (q - D)f(D)dD + b \int_q^{+\infty} (D - q) f(D)dD](\theta_l + \theta_h) + I_{Q_1 < Q_0}(e(q - Q_1)\theta_l + e(q - Q_h)\theta_h) + \alpha W[\theta_l p(\cdot | Q_t) + \theta_h p(\cdot | Q_h)] \right\}.
$$

(49)

where $\bar{W}(\rho)$ denotes the value function for the non-update model. Then we have

**Lemma 4.2:** The value function $W(\rho)$ for the partial-observation model is no greater than the value function $\bar{W}(\rho)$ for the non-update model.

**Proof:** Note that the uniqueness of $W(\rho)$ is proved in Theorem 3.1. We use the mapping $T$ defined in equation (15) and prove the lemma by value iteration according to $W^{n+1} = TW^n$. Similarly, we can define $\bar{W}^{n+1} = T\bar{W}^n$ and use the value iteration to obtain $\bar{W}(\rho)$ according to Equation (49). Let $W^0(\rho) = \bar{W}^0(\rho) = 0$. Suppose $W^n(\rho) \leq W^n(\rho)$ and by Equation (25) we have

$$
W^{n+1}(\rho) = \min_{q \geq 0} \left\{ [cq - h \int_0^q (q - D)f(D)dD + b \int_q^{+\infty} (D - q) f(D)dD](\theta_l + \theta_h) + I_{Q_1 < Q_0}(e(q - Q_1)\theta_l + e(q - Q_h)\theta_h) + \alpha W[\theta_l p(\cdot | Q_t) + \theta_h p(\cdot | Q_h)] \right\}
$$

$$
\leq \min_{q \geq 0} \left\{ [cq - h \int_0^q (q - D)f(D)dD + b \int_q^{+\infty} (D - q) f(D)dD](\theta_l + 1) + I_{Q_1 < Q_0}(e(q - Q_1)\theta_l + e(q - Q_h)\theta_h) + \alpha(\theta(1 - \beta_l) + \beta_h) \bar{W}(\rho) \cdot \left( \frac{\theta \beta_l + (1 - \beta_h)}{\theta (1 - \beta_l) + \beta_h} \right) \right\}.
$$

(51)

where the first inequality is by Lemma 4.1. Since $W^n(\rho)$ and $W^{n+1}(\rho)$ converges to $W(\rho)$ and $\bar{W}(\rho)$, respectively, we have the desired result. $\square$

To facilitate the numerical experiment, similarly to Equation (41), denote

$$
\bar{\Phi}(\rho) = \min_{q \geq 0} \left\{ [cq - h \int_0^q (q - D)f(D)dD + b \int_q^{+\infty} (D - q) f(D)dD](\theta_l + 1) + I_{Q_1 < Q_0}(e(q - Q_1)\theta_l + e(q - Q_h)\theta_h) + \alpha(\theta(1 - \beta_l) + \beta_h) \bar{W}(\rho) \cdot \left( \frac{\theta \beta_l + (1 - \beta_h)}{\theta (1 - \beta_l) + \beta_h} \right) \right\}.
$$

By the same value iteration approach, we can obtain the cost for different $\theta$ in Table I, where the improvement rate equals to $(\bar{\Phi}(\theta) - \Phi(\theta))/\Phi(\theta)$. Similarly, we can obtain Table II and III for $\beta = 1/3$ and $\beta = 1/2$, respectively. Note that $\beta < 0.5$ indicates that there is a greater possibility for the supply capacity to transit to the other state in the next period.

From Table I, II and III, we know that the cost saving for the partial-observation model is more sensitive to the transition probability $\beta$ than $\theta$ (representing the distribution of the current period supply capacity). Moreover, the cost saving can be significant when $\beta$ deviates from 0.5 and

### Table I

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>Partial-observation model $q^*$</th>
<th>Non-update Model $\Phi$</th>
<th>Improvement Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>416.8</td>
<td>698.4</td>
<td>40.3%</td>
</tr>
<tr>
<td>1</td>
<td>547.6</td>
<td>842.3</td>
<td>35.0%</td>
</tr>
<tr>
<td>2</td>
<td>756.8</td>
<td>1130.1</td>
<td>33.0%</td>
</tr>
</tbody>
</table>

### Table II

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>Partial-observation model $q^*$</th>
<th>Non-update Model $\Phi$</th>
<th>Improvement Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>348.0</td>
<td>565.2</td>
<td>38.4%</td>
</tr>
<tr>
<td>1</td>
<td>499.4</td>
<td>842.5</td>
<td>40.7%</td>
</tr>
<tr>
<td>2</td>
<td>741.9</td>
<td>1397.2</td>
<td>47.0%</td>
</tr>
</tbody>
</table>
TABLE III
COMPARISON WITH NON-UPDATE MODEL FOR DIFFERENT $\theta$, $\beta = 1/2$

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>Partial-observation model</th>
<th>Non-update model</th>
<th>Improvement Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>631.8 10.43</td>
<td>631.8 10.43</td>
<td>0%</td>
</tr>
<tr>
<td>1</td>
<td>842.4 10.43</td>
<td>842.4 10.43</td>
<td>0%</td>
</tr>
<tr>
<td>2</td>
<td>1263.6 10.43</td>
<td>1263.6 10.43</td>
<td>0%</td>
</tr>
</tbody>
</table>

equals to 0 when $\beta = 0.5$. Because when the self-transition probabilities are $\beta_t = \beta_h = \beta = 0.5$, there exists no difference between the cost-to-go’s for the non-update model ($\beta(1-\beta_t)$ + $\beta_h(1-\beta_h)$) in Equation (51) and for the partial-observation model ($\beta(1-\beta_t) + \beta_h(1-\beta_h)$) in Equation (41)).

V. CONCLUSIONS AND FUTURE RESEARCH

We have studied a multi-period news-vendor problem with partially observed supply capacity constraint. This partial supply capacity observations make dynamic programming be in a space of probability distributions. The existence of optimal purchasing policy is provided. Moreover, we demonstrate that, in a two-state Markovian capacity model, purchasing more than the lower possible supply capacity leads to a reduction of the buyer’s cost-to-go function. The buyer needs to balance a tradeoff between cost-to-go and current period cost savings when these two conflict with each other. Further we study the performance of the partial-observation model and the non-update model with and without dynamic updating supply capacity forecast based on the partially observed supply capacity information, respectively; we suggest the inventory managers pay a particular attention to the partially observed supply capacity information when there exists a supply capacity uncertainty.

Our future research will focus on studying the multi-period inventory decisions with partially observed supply lead time. And we also plan to consider the case when leftover inventory can be carried over to the next period instead of lost sales. This case would be much more complicated as it would bring the inventory level as the additional state variable.

APPENDIX I
PROOF OF LEMMA 3.2:

With equation (7), we know $W(\rho) \geq 0$. Since $W(\rho) = \inf_{q \geq 0} G(q) \geq G(0)$, where $G(q)$ is defined in equation (7), and $L(D, 0) \leq bD$, we can obtain that

$$W(\rho) \leq \int L(D, 0) f(D) dD \cdot \int \rho(x) dx + \alpha W\left(\int_0^\infty \rho(\xi)p(\cdot | \xi) d\xi\right)$$

$$\leq b \int D f(D) dD \cdot \int \frac{x \rho(x) dx}{\pi(x) dx} + \alpha \|W||_{B} \cdot \int_0^\infty \rho(\xi)p(\cdot | \xi) d\xi$$

$$\leq \frac{b \mu_D}{\mu_Q} \int x \rho(x) dx + \alpha \|W||_{B} \int \rho(\xi) d\xi + c_0 \int x \rho(x) dx$$

$$\leq \frac{b \mu_D}{\mu_Q} \|\rho\| + \alpha \max\{1, c_0\} \|W||_{B} \|\rho\|, \quad (52)$$

where the third inequality is due to the following facts:

$$\mu_D := \int D f(D) dD, \quad \mu_Q := \int Q(\pi(Q) dQ, \quad (53)$$

and

$$\|\int_0^\infty \rho(\xi)p(\cdot | \xi) d\xi\| = \int \int p(x|\xi)\rho(\xi) d\xi dx$$

$$+ \int x \int p(x|\xi)\rho(\xi) d\xi dx$$

$$\leq \int \rho(\xi) d\xi + c_0 \int \xi \rho(\xi) d\xi, \quad (54)$$

which is due to equation (10) and assumption 3.1.

Hence, by the definition of norm in space $B$, from equation (52), we can obtain that

$$\|W||_{B} \leq \frac{b \mu_D}{\mu_Q} + \alpha \max\{1, c_0\} \|W||_{B}, \quad (55)$$

so that

$$\|W||_{B} \leq \frac{b \mu_D}{\mu_Q(1-\alpha \max\{1, c_0\})} := k_0. \quad (56)$$

Recalling $\alpha c_0 < 1$, we know $k_0 > 0$. Hence $W \in B$. \quad \Box

APPENDIX II
PROOF OF LEMMA 3.2:

Fix $q$ and define

$$T_q(W)(\rho) := \int L(D, q) f(D) dD \int \rho(x) dx$$

$$+ \int_0^q e(q - Q)\rho(Q) dQ$$

$$+ \alpha \int_0^q W(p(\cdot | Q))\rho(Q) dQ$$

$$+ \alpha W\left(\int_0^{+\infty} \rho(\xi)p(\cdot | \xi) d\xi\right). \quad (57)$$
Then we can obtain
\[
|T_q(W)(\rho) - T_q(\tilde{W})(\rho)| \\
= |\alpha \{ W(\int_0^{+\infty} \rho(\xi)p(\cdot | \xi)d\xi) \\
- \tilde{W}(\int_0^{+\infty} \rho(\xi)p(\cdot | \xi)d\xi) \} \\
+ \alpha \left\{ \int_0^{q} (W(p(\cdot | Q)) - \tilde{W}(p(\cdot | Q)))\rho(Q)dQ \right\}| \\
\leq \alpha \|W - \tilde{W}\|_B \cdot \| \int_0^{+\infty} \rho(\xi)p(\cdot | \xi)d\xi\| \\
+ \alpha \|W - \tilde{W}\|_B \int_0^q \|p(\cdot | Q)\|\rho(Q)dQ \\
\leq \alpha \|W - \tilde{W}\|_B \left\{ \int_0^{+\infty} \rho(\xi)d\xi \\
+ \int x \int_0^{+\infty} p(x | \xi)\rho(\xi)d\xi dx \\
+ \int_0^q (1 + \int xp(x | \xi)dx)\rho(\xi)d\xi \right\} \\
\leq \alpha \|W - \tilde{W}\|_B \left\{ \int \rho(\xi)d\xi + \int \xi \rho(\xi)d\xi \right\} \\
\leq \alpha \max\{1, c_0\} \|W - \tilde{W}\|_B \left\{ \int \rho(\xi)d\xi + \int \xi \rho(\xi)d\xi \right\}. \\
\]
\[\tag{58}\]

Without loss of generality, assume \(\min_{q \geq 0}\{T_q(W)(\rho)\} = T_{q^*}(W)(\rho)\geq \min_{q \geq 0}\{T_q(\tilde{W})(\rho)\} = T_{q^*}((\tilde{W})(\rho)\); hence
\[
|\min\{T_q(W)(\rho)\} - \min\{T_q(\tilde{W})(\rho)\}| \\
= |T_{q^*}(W)(\rho) - T_{q^*}(\tilde{W})(\rho)| \\
\leq |T_{q^*}(W)(\rho) - T_{q^*}(\tilde{W})(\rho)| \\
\leq |T_{q^*}(W)(\rho) - T_{q^*}(\tilde{W})(\rho)| \\
\leq \alpha \max\{1, c_0\} \|W - \tilde{W}\|_B \left\{ \int \rho(\xi)d\xi + \int \xi \rho(\xi)d\xi \right\}, \\
\]
\[\tag{59}\]
i.e.
\[
|T(W)(\rho) - T(\tilde{W})(\rho)| \leq \alpha \max\{1, c_0\} \|W - \tilde{W}\|_B \cdot \|\rho\|. \\
\]
\[\tag{60}\]

Hence
\[
|T(W) - T(\tilde{W})| \\
\leq \alpha \max\{1, c_0\} \|W - \tilde{W}\|_B. \\
\]
\[\tag{61}\]

**APPENDIX**

**PROOF OF LEMMA 3.3:**

Let \(W_B^0(\rho) = 0, \rho \in \mathcal{H}^+,\) and define, recursively
\[
W_B^{n+1}(\rho) = \min_{q \in [0, B]} \left\{ \int L(D, q)f(D)dD \int \rho(x)dx \\
+ \int_0^q e(q - Q)\rho(Q)dQ \\
+ e \int_0^q W_B(p(\cdot | Q))\rho(Q)dQ \right\}. \\
\]

Step 1) It is obviously that the lemma holds for \(W_B^0(\rho);\)
Step 2) Suppose it holds for \(W_B^n(\rho),\) i.e.
\[
|W_B^n(\rho) - W_B^n(\tilde{\rho})| \leq H_B \|\rho - \tilde{\rho}\|. \\
\]
we prove that it holds for \(W_B^{n+1}(\rho).\) Since
\[
\int L(D, q)f(D)dD \int \rho(x)dx \\
+ e \int_0^q (q - Q)\rho(Q)dQ \\
\leq \max\{b\mu_D, cq\} \int \rho(x)dx \\
+ e \int_0^q (q - Q)\rho(Q)dQ \\
\leq \max\{b\mu_D, cB, eB\} \left\{ \int \rho(x)dx \\
+ \int Q\rho(Q)dQ \\
\right\} \\
= \max\{b\mu_D, cB, eB\} \|\rho - \tilde{\rho}\|, \\
\]
where the first inequality is due to \(L(D, q) \leq \max\{cq, BD\}\) and the second inequality is due to \(q \leq B;\)
\[
|\int_0^q W_B^n(p(\cdot | Q))\rho(Q)dQ - \int_0^q W_B^n(p(\cdot | Q))\tilde{\rho}(Q)dQ| \\
= \int_0^q W_B^n(p(\cdot | Q))\rho(Q)dQ - \int_0^q W_B^n(p(\cdot | Q))\tilde{\rho}(Q)dQ \\
\leq k_0 \left\{ \int \rho(Q) - \tilde{Q}dQ + \int Qp(x|Q)\rho(Q) - \tilde{Q}dQ \right\} \\
\leq k_0 \max\{1, c_0\} \|\rho - \tilde{\rho}\|. \\
\]
\[\tag{65}\]
and
\[\begin{align*}
|W_B^n| & \int_q^{+\infty} \rho(\xi)p(\cdot | \xi)d\xi \\
- & W_B^n(\int_q^{+\infty} \tilde{\rho}(\xi)p(\cdot | \xi)d\xi) \\
\leq & H_B \left\{ \int_q^{+\infty} |\rho(\xi) - \tilde{\rho}(\xi)|p(x| \xi)d\xi \right\} \\
+ & \int_q^{+\infty} |\rho(\xi) - \tilde{\rho}(\xi)|p(x| \xi)d\xi \\
\leq & H_B \{ \int_q^{+\infty} |\rho(\xi) - \tilde{\rho}(\xi)|d\xi \\
+ & c_0 \int_0^{+\infty} |\xi(\rho(\xi) - \tilde{\rho}(\xi))|d\xi \\
\leq & H_B \max\{1, c_0\}||\rho - \tilde{\rho}||. 
\end{align*}\]

Combining equations (64), (65) and (66), we can obtain
\[\begin{align*}
|W_B^{n+1}(\rho) - W_B^{n+1}(\tilde{\rho})| \\
\leq & (\max\{b\mu_D, e\alpha B, eB, e\} + \alpha k_0 \max\{1, c_0\})||\rho - \tilde{\rho}||. 
\end{align*}\]

Now we only need to let the right hand-side of (67) is smaller than or equal to $H_B||\rho - \tilde{\rho}||$. This is true when
\[H_B := \frac{\max\{b\mu_D, e\alpha B, eB, e\} + \alpha k_0 \max\{1, c_0\}}{1 - \alpha \max\{1, c_0\}}.\]

Then we obtain
\[|W_B^{n+1}(\rho) - W_B^{n+1}(\tilde{\rho})| \leq H_B||\rho - \tilde{\rho}||.\]

Together with Step 1), we can finish the proof. \qed

REFERENCES


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