Dynamic Pricing Control for Open Queueing Networks

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Abstract—Pricing control is an important problem in service systems and it aims to control customer behaviors through an economic way, instead of administrative commands. In this paper, we study a dynamic pricing and service rate control problem in an open Jackson network with limited capacity. The goal is to determine the optimal admission prices and the optimal service rates at every state such that the long-run average social welfare is maximized. The original problem is decomposed into a rate-setting problem plus a price-setting problem. To solve the rate-setting problem, we derive a difference formula based on the sensitivity-based optimization theory. When the cost rate function is convex in service rates and the value rate function is concave in arrival rates, we decompose the rate-setting problem into a series of convex optimization subproblems. When the rate functions have linear structure, these subproblems are even simpler and a bang–bang control is optimal. For the price-setting problem, we determine the state-dependent prices so as to induce the optimal arrival rates obtained by the rate-setting problem. We propose a recursive algorithm to numerically compute the conditional expected delays at every state. Finally, we conduct numerical experiments to explore the optimality properties and some useful insights for this dynamic pricing control problem.

Index Terms—Admission control, pricing control, queueing networks, sensitivity-based optimization, service rate control.

I. INTRODUCTION

Pricing control is an important way to affect customer behaviors in service systems. Adjusting the price dynamically has been widely adopted in many business activities. For instance, in power industry, a higher price is charged on users during the peak of electricity consumption in daytime, whereas the price drops along with the decrease in the amount of electricity consumption at night. An Internet service provider charges users a floating fee depending on the current number of users. Other examples widely exist in the pricing of airline tickets, congestion tolls, etc. [1], [2]. In addition to dynamic pricing, allocating service resources dynamically is also widely adopted by service operators, with the merit of keeping the expenditure relatively low while meeting the demand of customers.

In the literature on queueing theory, there are extensive works concerning the pricing and service rate control. The detailed research review on pricing control for queueing models can be referred to [3]. Some pioneering works consider the static pricing scheme with the objective of maximizing social welfare or revenue of the system in an M/M/1 queue [4]. With the obvious advantages provided by the dynamic pricing strategy, dynamic pricing has gained more and more research attention [5]. Ata and Shneorson [6] and Low [7] use dynamic programming to study optimal state-dependent prices in an M/M/1 and an M/M/s queue, respectively. When it comes to queueing networks, it is difficult to apply the classical dynamic programming approach due to the high dimensionality of resulting optimization problems. Adelman [8] uses the approximate dynamic programming to study the near-optimal state-dependent pricing policy in a closed logistics queueing network. Banerjee et al. [9] employ a large-scale market limit to study the static pricing and threshold dynamic pricing problem in a two-sided ride-sharing platform that operates over a network of regions. They find that even though the performance under any dynamic pricing policy cannot exceed that under the optimal static pricing policy, the dynamic pricing is much more robust to fluctuations in system parameters compared to the static pricing. Masuda and Whang [10] study the dynamic pricing scheme based on actual or estimated arrival rates due to lack of demand information in a data communication network. So far, the problem of optimal state-dependent pricing has not been completely solved for queueing networks in the literature.

The second line of related research focuses on the dynamic service rate control in queueing systems. There is a series of works studying the structure of optimal service rates with the objective of minimal time-average cost. For single-server queueing systems, the optimality of threshold type policy, such as $N$-policy or $D$-policy, has been extensively studied [11], [12]. For multiple-server queueing networks, such as cyclic queues or tandem queues, the monotone structure of optimal service rates is also studied [13], [14]. For complicated Jackson networks, it is proved that when the cost function is linear in the service rate, the optimal policy is of a bang–bang control form [15], [16]. Recent work further extends the optimality of bang–bang
control from linear case to concave case [17]. The optimal state-dependent and load-dependent service rates in closed Jackson networks have been solved in [18] and [19].

Queueing networks are a useful model to study the system performance of ride-sharing or vehicle-sharing platforms [9], [20], [21], logistics networks [8], communication networks [2], [10], etc. Investigating the dynamic pricing and control problem in a queueing network is of great challenge since it is far more complex than solving that in a single-server queueing system. In this paper, we study an open Jackson network, where the service provider posts the admission price and the expected delay at each system state. Customers, who are delay-sensitive and rational decision makers, observe the posted information upon arrival and decide whether to enter the system based on their own net utility. The objective of the service provider is to find optimal state-dependent prices and service rates such that the long-run average social welfare is maximized. For the case where customers are heterogeneous in terms of service value, which is a classical model proposed by Mendelson [1], the marginal service value captures the relationship between the price and the effective arrival rate. Thus, the original problem can be decomposed into a rate setting problem and a price setting problem, which is also the core idea of the decomposition technique used in this paper. We first solve the optimal state-dependent arrival and service rates via an iterative algorithm developed based on performance difference formulas, without consideration of the customer behaviors. Then, we utilize the optimal rates obtained above to solve the optimal state-dependent prices and expected delays that coordinate the system to achieve the same optimality when the customer behaviors are considered. We develop a recursive algorithm to numerically compute the expected delay of customers throughout the network, at the condition that the system state is given upon customer arrivals.

The contributions of this paper are threefold.

1) We solve the dynamic pricing and service rate control problem for open Jackson networks. To our knowledge, the dynamic pricing and control problem with delay-sensitive and rational customers has not been studied for queueing networks, whereas most of the previous works focus on simple queueing models.

2) We propose a recursive algorithm to compute the conditional expected delay in open Jackson networks, where arrival and service rates are both state-dependent. Such algorithm does not appear in the literature.

3) We demonstrate that when the value rate function is concave in the arrival rate and the cost rate function is convex in the service rate, the optimal state-dependent arrival and service rates are solved by a series of convex optimization subproblems. Furthermore, if the rate functions have linear structures, these subproblems are more straightforward: the optimal state-dependent arrival and service rates are either the maximum or the minimum of value domains and the optimal policy is of a bang–bang control form.

The remainder of this paper is outlined as follows. In Section II, we give a mathematical formulation for the dynamic pricing and control problem in queueing networks. In Section III, we first prove that the original pricing problem is equivalent to a rate setting problem plus a price setting problem. Then, we solve these problems separately. The optimality properties and algorithms are also proposed in this section. In Section IV, we conduct numerical experiments to demonstrate the main results. Finally, we conclude this paper in Section V.

II. PROBLEM FORMULATION

Consider the dynamic pricing control in a service facility network, which is illustrated in Fig. 1. The system is modeled as an open Jackson network with $M$ servers. The network has a finite capacity $N$. Note that this capacity constraint can be loosened since it is also a consequence of rational customer behaviors, as we will discuss later in Remark 1. An open Jackson network with a finite capacity is also known as a semiopen Jackson network in which the external arrival process can be viewed as a virtual server indexed as 0 [22]. Customers joining the network are transferred among servers according to routing probabilities: customers entering the network are allocated to server $i$ with probability $q_{0i}$, $i = 1, \ldots, M$ and $\sum_{i=1}^{M} q_{0i} = 1$; after service completion at server $i$, customers will transit to server $j$ with probability $q_{ij}$, $i, j = 1, 2, \ldots, M$; customers leave the network from server $i$ with probability $q_{0i}$, $i = 1, 2, \ldots, M$. In order to avoid an infinite loop of customers, we assume $(I - Q)^{-1}$ exists where $Q$ is the $M \times M$ routing probability matrix whose $i$-row $j$-column element is the routing probability $q_{ij}$. We have $\sum_{j=0}^{M} q_{ij} = 1$ for $i = 1, \ldots, M$. Without loss of generality, we assume $q_{ii} = 0$ for $i = 0, \ldots, M$. The service discipline is first come first serve. When a server is busy, newly arriving customers will wait in the buffer of that server. The buffer size is adequate. The number of customers at server $i$ (including both being served and waiting to be served) is denoted as $n_{i}$, $i = 1, \ldots, M$. The number of customers at server 0 is defined as $n_{0} = N - \sum_{i=1}^{M} n_{i}$. The state of the network
is denoted as \( n := (n_1, \ldots, n_M) \). The state space is defined as 
\[ S := \{ n : \sum_{i=1}^{M} n_i \leq N \}, \] 
whose size is \( |S| = \binom{M+N}{N} \), where
\( |S| \) represents the cardinality of set \( S \).

The service time of server \( i \) at state \( n \) is exponentially distributed with a state-dependent service rate \( \mu_i, n \) chosen from value domain \([0, U]\). The external customer arrival follows a Poisson process with a fixed rate \( \Lambda \). Customers receiving service will gain a service value \( u \) that is a positive continuous random variable drawn from a given distribution \( F \). We assume that \( u \in [\underline{u}, \overline{u}] \) and the associated probability density distribution (p.d.f.) is denoted as \( f \). We further assume that \( f(u) > 0 \) for \( u \in [\underline{u}, \overline{u}] \). Customers are delay sensitive: each customer expects delay cost \( v \) for one unit of its sojourn time in the system. At each state \( n \), the service provider posts an expected delay \( D_n \) and a price \( p_n \) charged for service. Note that \( p_n \) can be negative and it has a lower bound \( \underline{p} \), i.e., \( p_n \geq \underline{p} \). Customers upon arrival, observing the posted information, decide whether or not to enter the system based on their own expected net utility.

The expected net utility equals the gained service value minus the charged price and the expected delay cost, which is written as follows:
\[ R_n = u - p_n - v \cdot D_n. \]  

We assume that every customer is a rational decision maker. That is, if \( R_n > 0 \), this customer will enter the system to request service. Otherwise, it will not enter the system. Therefore, we can further define the threshold service value \( \tilde{u}_n \) as follows:
\[ \tilde{u}_n := \min \{ \max \{ p_n + v \cdot D_n, u \}, \overline{u} \} \]  

At each state \( n \), only customers with a service value larger than \( \tilde{u}_n \) will enter the system. The probability that customers enter the system at state \( n \) is \( 1 - F(\tilde{u}_n) \). Consequently, the effective arrival rate at state \( n \) is
\[ \lambda_n = \Lambda(1 - F(\tilde{u}_n)). \]

Note that when \( p_n \leq u - v \cdot D_n, \tilde{u}_n = u \), and \( \lambda_n = \Lambda \), i.e., all customers enter the system. Similarly, when \( p_n \geq \overline{u} - v \cdot D_n, \tilde{u}_n = \overline{u} \), and \( \lambda_n = 0 \), i.e., all customers balk.

Let \( F^{-1} \) be the inverse function of the distribution function \( F \). We define \( F^{-1}(0) = \underline{u} \) and \( F^{-1}(1) = \overline{u} \), then the threshold service value \( \tilde{u}_n \) can also be written as follows:
\[ \tilde{u}_n = F^{-1} \left( 1 - \frac{p_n}{\Lambda} \right). \]

Fig. 2 is an illustrative example in which the service value \( u \) obeys a uniform distribution in \( [\underline{u}, \overline{u}] \). The customers whose service value drawn in the shaded area will enter the system to request service.

Since only the customers with positive \( R_n \) at state \( n \) will enter the system, the service value generated by these admitted customers is a new random variable that obeys the p.d.f. \( \frac{f(v)}{1 - F(\tilde{u}_n)} \) in the interval \( [\tilde{u}_n, \overline{u}] \), where \( \underline{u} \leq \tilde{u}_n < \overline{u} \). Note that if \( \tilde{u}_n = \overline{u} \), no customer joins the system and service values generated by the system is zero. When \( \underline{u} \leq \tilde{u}_n < \overline{u} \), the system total service values generated per unit of time at state \( n \) can be written as
\[ b(n) = \lambda_n \int_{x=\underline{u}}^{\overline{u}} \frac{f(x)}{1 - F(\tilde{u}_n)} dx. \]

To operate service at rate \( \mu_i, n \), the service provider has to pay an operating cost \( c(\mu_i, n) \) per unit of time. It is assumed that \( c(\mu_i, n) \) is convex and nondecreasing in \( \mu_i, n \) where \( \mu_i, n \in [0, U] \). Other than the operating cost, the system also incurs a holding cost. The holding cost rate reflects the delay cost caused by the waiting time of customers in one unit of time.

In this paper, we assume the holding cost rate as \( v \cdot \sum_{i=1}^{M} n_i \), where \( v \) is a given coefficient. The social welfare of the whole system is defined as the social utility of customers minus the cost of operating the system. Based on this definition, the social welfare function at state \( n \) can be written as
\[ f(n) = b(n) - \sum_{i=1}^{M} c(\mu_i, n) - v \cdot \sum_{i=1}^{M} n_i. \]

We further denote \( f \) as a \( |S| \)-dimensional column vector composed of element \( f(n) \), where \( n \in S \).

Let \( n_t \) indicate the system state at time \( t \). The long-run average social welfare of the whole system can be written as
\[ \eta = \mathbb{E} \{ f(n_t) \} = \sum_{n \in S} \pi(n) \cdot f(n) = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f(n_t) dt \]

where \( \pi(n) \) is the stationary probability of the system staying at \( n \) and the last equality is valid for ergodic case.

Remark 1. Such a price controlled queueing network is always stable since the system has a capacity constraint \( N \). If \( N \to \infty \), the mechanism of rational customer behaviors has a self-tuning effect that stabilizes the network: when the number of customers \( |n| \) is large enough, the expected delay \( D_n \) is also large. Since the price \( p_n \) has a lower bound \( \underline{p} \) and \( v > 0 \), if we have \( p + v \cdot D_n > \overline{u} \), no more customer will enter the network. The network dynamics behave as if it has a capacity constraint and the network is always stable.

With the above-mentioned formulation, this dynamic pricing and control problem can be stated as follows.
Problem 0: The dynamic pricing and control problem is to choose optimal state-dependent prices and service rates such that the long-run average social welfare is maximal. Thus, Problem 0 is denoted as
\[
P_0: \max_{\lambda, \mu} \eta(\lambda, \mu)
\]
where \(\lambda := (p_n : p_n > \mu, n \in S), \mu := (\mu_i : \mu_i \in [0, U], i = 1, \ldots, M, n \in S), \) and \(\lambda := (\lambda_n, \lambda_n \in [0, \Lambda], n \in S).\) The optimal average social welfare for P0 is denoted as \(\eta_{P0}.\)

In (5), the service value rate function depends on the threshold \(\lambda_n,\) which in turn depends on the price \(p_n\) and the expected delay \(D_n.\) Expected delay \(D_n\) has a complicated connection with \(p\) and \(\mu,\) which makes Problem 0 difficult to solve. Moreover, from (5) and (7), taking the expectation over the integral term in \(b(n)\) makes the problem even more complicated. Therefore, it is difficult to solve Problem 0 directly.

Below, we resort to an alternative way to express the service value rate function. Given (4), we see that \(b\) can also be viewed as a function of \(\lambda_n.\) Therefore, we denote it as \(b(\lambda_n)\) for simplicity.

Substituting (4) into (5), we have
\[
b(\lambda_n) = \lambda_n \cdot \int_{\bar{u}_n}^{x} \cdot \frac{1}{1 - F(u)} dx
\]
\[
= \Lambda \cdot \int_{F^{-1}(1 - \frac{\mu}{\Lambda})}^{x} \cdot \frac{1}{1 - F(u)} dx.
\]

Taking the derivative with respect to \(\lambda_n,\) we have
\[
b'(\lambda_n) = -\Lambda \cdot F^{-1}
\]
\[
\cdot \int_{\frac{\lambda_n}{\Lambda}}^{x} \cdot \frac{1}{1 - F(u)} dx.
\]

The physical meaning of \(b'(\lambda_n)\) is the marginal service value at state \(n,\) that is, the change in the aggregate service value rate of total customers induced by a unit increase in the arrival rate \(\lambda_n.\) It is interesting to find that \(b'(\lambda_n)\) exactly matches threshold service value \(\bar{u}_n\) defined in (4). Besides, since \(F^{-1}(1 - \frac{\mu}{\Lambda}) \in [\mu, \bar{u}] > 0, b(\lambda_n)\) is increasing in \(\lambda_n \in [0, \Lambda].\) Given that \(F(u)\) is continuous and monotone increasing in \(u \in [\mu, \bar{u}],\) we have
\[
b''(\lambda_n) = -\frac{1}{\Lambda} \cdot \frac{1}{\int_{F^{-1}(1 - \frac{\lambda_n}{\Lambda})}^{x} \cdot \frac{1}{1 - F(u)} dx}.
\]

Since \(b''(\lambda_n) < 0, b(\lambda_n)\) is strictly concave in \(\lambda_n \in [0, \Lambda].\)

Based on our discussion in Section II, the price \(p_n\) that induces effective arrival rate \(\lambda_n\) can be computed by
\[
p_n = F^{-1}(1 - \lambda_n/\Lambda) - v \cdot D_n.
\]

Note that for arrival rate \(\lambda_n = 0,\) the price that induces it can be any value equal to or larger than \(\bar{u} - v \cdot D_n,\) and for arrival rate \(\lambda_n = \Lambda,\) the price that induces it can be any value equal to or smaller than \(\mu - v \cdot D_n.\)

Equation (12) provides the intuitive way to solve Problem 0: We can first solve the optimal effective arrival rates \(\lambda_n^*\) and service rates \(\mu_n^*\) that maximize the long-run average welfare, then we can determine the corresponding optimal prices \(p_n^*\) and service rates \(\mu_n^*\) are obtained. Based on this idea, we decompose Problem 0 into two problems, i.e., the rate-setting problem and the price-setting problem. The complete statements of these two problems are described as follows.

Problem 1: The rate-setting problem is to choose the optimal state-dependent arrival and service rates that maximize the long-run average welfare, without considering the customer behavior. Here, the welfare rate function is defined as
\[
f(n) = b(\lambda_n) - \sum_{i=1}^{M} c(\mu_i, n) - v \cdot \sum_{i=1}^{M} n_i.
\]

Problem 1 is denoted as
\[
P1: \max_{\lambda, \mu} \eta(\lambda, \mu).
\]

The optimal average social welfare for P1 is denoted as \(\eta_{P1}.\)

Problem 2: The price-setting problem is to determine the state-dependent prices such that the long-run average welfare generated via the pricing strategy is equal to the one obtained in the rate-setting problem. In this problem, the service rates are the same as the optimal ones in Problem 1. Problem 2 is denoted as
\[
P2(\lambda, \mu): \{p : \eta(p, \mu) = \eta(\lambda, \mu)\}.
\]

III. MAIN RESULTS

Before proceeding further, we need to prove that the dynamic pricing and control problem is equivalent to the rate-setting problem combined with the price-setting problem formulated above. We have the following theorem to characterize the relation of the three problems P0, P1, and P2.

Theorem 1: For any optimal solution \((\lambda^*, \mu^*)\) to P1 and \((\lambda^*, \mu^*)\) to P2, \((\lambda^*, \mu^*)\) is also the optimal solution to P0 and \(\eta_{P0} = \eta_{P1}.\)

Proof: First, we prove that \(\eta_{P0} = \eta_{P1}.\) For any policy \((\lambda, \mu)\) feasible for P1, the arrival rates \(\lambda\) can be induced by the prices \(p\) given in (12), as can be seen by (2) and (4). We can see that \((p, \mu)\) is always feasible for P0, so \(\eta_{P0} \geq \eta_{P1}.\) On the other hand, any \((p, \mu)\) feasible for P0 induces arrival rates \(\lambda\) given by (3) that are also feasible for P1, so \(\eta_{P1} \geq \eta_{P0}.\) Therefore, we have \(\eta_{P0} = \eta_{P1}.\)

Second, we prove that \((\lambda^*, \mu^*)\) is the optimal solution to P0. Since \((\lambda^*, \mu^*)\) is the optimal solution to P1, we have \(\eta(\lambda^*, \mu^*) = \eta_{P1}.\) Since \(p^*\) is the solution to P2, we have \(\eta(p^*, \mu^*) = \eta(\lambda^*, \mu^*).\) Therefore, we have \(\eta(p^*, \mu^*) = \eta_{P1}.\) As we have already proved \(\eta_{P0} = \eta_{P1}\) in the above paragraph, we have \(\eta(\lambda^*, \mu^*) = \eta_{P0}.\) Therefore, from the definition of P0, we know that \((\lambda^*, \mu^*)\) is the optimal solution to P0.
A. Rate-Setting Problem

In this subsection, we use the sensitivity-based optimization theory [23] to study Problem 1. For the Markov chain \( X := \{x_i, t \geq 0 \} \) formulated in Section II, its infinitesimal generator is denoted as a matrix \( B \). The steady-state distribution is denoted as a row vector \( \pi \) whose element is \( \pi(n), n \in S \). We have \( B = 0, \pi B = 0 \), and \( \pi e = 1 \), where \( e \) is a \(|S|\)-dimensional column vector with all elements as 1.

In Markov systems, the perturbation of a policy is reflected in the infinitesimal generator \( B \) and the reward function \( f \). We denote \( B', f', \pi' \), and \( \eta' \) as the infinitesimal generator, the reward function, the steady-state distribution, and the long-run average performance corresponding to a new policy \((\lambda', \mu')\), respectively. The difference of the system average performance caused by the perturbation of policies can be written as follows [23]:

\[
\eta' - \eta = \pi'[(B' - B)g + (f' - f)]
\]  

where \( g \) is referred to as the performance potential of the Markov system under \( B \) and \( f, g \) is a \(|S|\)-dimensional column vector, whose definition can be found in (27).

We denote the neighboring states of the system state \( n \) as \( n_{i+j} := (n_1, \ldots, n_i + 1, \ldots, n_M) \), \( n_{i-} := (n_1, \ldots, n_i - 1, \ldots, n_M) \), and \( n_{i-} := (n_1, \ldots, n_i - 1, \ldots, n_j + 1, \ldots , n_M) \), where \( 1 \leq i, j \leq M \). The elements of the infinitesimal generator \( B \) are given as follows [18]:

\[
\begin{align*}
\sum_{i=1}^{M} q_{i0}[g(n_{i+1}) - g(n)] &+ b(\lambda_n) \\
\sum_{i=1}^{M} q_{0i}[g(n_{i-1}) - g(n)] &+ b(\lambda_n')
\end{align*}
\]

\[
\eta' - \eta = \sum_{n \in S} \pi(n) \left\{ \left[ (\lambda_n' - \lambda_n) \sum_{i=1}^{M} q_{i0}[g(n_{i+1}) - g(n)] + b(\lambda_n') - b(\lambda_n) \right] \right\}.
\]  

(17)

Similarly, we can also derive the performance difference resulting from the change of service rates from \( \mu_i, n \) to \( \mu_i', n \), \( i = 1, \ldots, M \) and \( n \in S \)

\[
\eta' - \eta = \sum_{n \in S} \pi'(n) \left\{ \sum_{i=1}^{M} (\mu_i', n - \mu_i, n) \sum_{j=0}^{M} q_{ij} \right\}
\]

\[
[g(n_{i-1}) - g(n)] + c(\mu_i', n) - c(\mu_i, n) \right\}.
\]  

(18)

Analyzing the performance difference formulas (17) and (18), we can derive the following theorem.

**Theorem 2:** If we choose a policy \((\lambda', \mu')\) that satisfies

\[
\lambda_n' \sum_{i=1}^{M} q_{i0}[g(n_{i+1}) - g(n)] + b(\lambda_n') \\
\geq \lambda_n \sum_{i=1}^{M} q_{i0}[g(n_{i+1}) - g(n)] + b(\lambda_n)
\]

\[
(19)
\]

and

\[
\mu_i' \sum_{j=0}^{M} q_{ij}[g(n_{i-1}) - g(n)] - c(\mu_i, n) \\
\geq \mu_i \sum_{j=0}^{M} q_{ij}[g(n_{i-1}) - g(n)] - c(\mu_i, n)
\]

\[
(20)
\]

\( i = 1, \ldots, M \), for all \( n \in S \), then we have \( \eta' \geq \eta \). If either the inequality (19) or (20) strictly holds for at least a state \( n \) with positive steady-state probability \( \pi'(n) \) under policy \((\lambda', \mu')\), then we have \( \eta' > \eta \).

The above theorem is easy to obtain from the performance difference formulas (17) and (18). We omit the proof for simplicity.

Based on Theorem 2, we can further obtain a necessary and sufficient condition for the optimal policy of Problem 1.

**Theorem 3:** A policy \((\lambda, \mu)\) is optimal if and only if

\[
\lambda_n \sum_{i=1}^{M} q_{i0}[g(n_{i+1}) - g(n)] + b(\lambda_n) \\
\geq \lambda_n' \sum_{i=1}^{M} q_{i0}[g(n_{i+1}) - g(n)] + b(\lambda_n')
\]

\[
(21)
\]

and

\[
\mu_i \sum_{j=0}^{M} q_{ij}[g(n_{i-1}) - g(n)] - c(\mu_i, n) \\
\geq \mu_i' \sum_{j=0}^{M} q_{ij}[g(n_{i-1}) - g(n)] - c(\mu_i, n)
\]

\[
(22)
\]

for all \((\lambda', \mu')\) and \( n \in S \).

**Proof:** The sufficient condition: If a policy \((\lambda, \mu)\) satisfies (21) and (22) for all \((\lambda', \mu')\), according to Theorem 2, \( \eta \geq \eta' \).

Thus, \((\lambda, \mu)\) is optimal.

The necessary condition: Suppose that \((\lambda, \mu)\) is the optimal policy but it does not satisfy (21) and (22). So there exists at least one policy, denoted by \((\lambda^b, \mu^b)\), for which (21) or (22) does not hold. In other words, there exists at least one state, denoted as \( n^* \), such that

\[
\lambda_n \sum_{i=1}^{M} q_{i0}[g(n_{i+1}) - g(n)] + b(\lambda_n) \\
< \lambda_n^b \sum_{i=1}^{M} q_{i0}[g(n_{i+1}) - g(n)] + b(\lambda_n^b)
\]

\[
(23)
\]

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or
\[
\mu_{i,n} \sum_{j=0}^{M} q_{ij} [g(n_{i,j}^t) - g(n)] - c(\mu_{i,n})
\]
\[
< \mu_{i,n}^{b} \sum_{j=0}^{M} q_{ij} [g(n_{i,j}^t) - g(n)] - c(\mu_{i,n}^{b})
\] (24)
holds. So, we can build a new policy \( (\lambda^d, \mu^d) \), where \( \lambda^d = \lambda^u \), \( \mu = \mu^u \), and for all \( n \neq n^t \), \( \lambda^d = \lambda_n^t \), \( \mu^d = \mu_n^t \).

According to Theorem 2, the long-run average performance \( \eta^d \)
under the policy \( (\lambda^d, \mu^d) \) has \( \eta^d > \eta \). This contradicts with the assumption that \( (\lambda, \mu) \) is the optimal policy. So if \( (\lambda, \mu) \) is optimal, it satisfies (21) and (22)

With Theorems 2 and 3, we can establish a policy improvement
step as the following subproblems:
\[
\lambda_n^t = \arg \max_{\lambda_n \in [0, \Lambda]} \left\{ \lambda_n \sum_{i=1}^{M} q_{0i} [g(n_{i+1}) - g(n)] + b(\lambda_n) \right\}, \quad n \in S
\] (25)
\[
\mu_n^t = \arg \max_{\mu_n \in [0, U]} \left\{ \mu_n \sum_{j=0}^{M} q_{ij} [g(n_{i,j+1}) - g(n)] - c(\mu_n) \right\}, \quad i = 1, \ldots, M, \quad n \in S
\] (26)
where \( g(n) \) is defined as
\[
g(n) = \lim_{T \to \infty} \mathbb{E} \left\{ \int_0^T [f(n_t) - \eta]dt \right\}
\] (27)
which can be computed by solving the following Poisson equation:
\[
(B - e\pi)g = -f.
\] (28)

According to Theorem 2, the updated arrival rate \( \lambda_n^t \) (or service rate \( \mu_n^t \)) produces a performance that is greater or at least equal to the performance under the current arrival rate \( \lambda_n \) (or service rate \( \mu_n \)).

To solve (25) and (26), we have to know the structure of functions \( b(\lambda_n) \) and \( c(\mu_n) \). Different function structures will affect the optimization complexity of (25) and (26), and determine the optimality property of optimal solution \( \lambda_n^* \) and \( \mu_n^* \). Below, we give some discussions according to the different forms of functions \( b(\lambda_n) \) and \( c(\mu_n) \).

**Case 1:** Given that the service value function \( b(\lambda_n) \) is strictly concave in the arrival rate \( \lambda_n \) and the operating cost rate function \( c(\mu_n) \) is convex in the service rate \( \mu_n \), the expressions in the brackets of (25) and (26) are concave in \( \lambda_n \) and \( \mu_n \), respectively. As a result, the subproblems (25) and (26) are convex optimization problems and the new policy \( \lambda_n^t \) and \( \mu_n^t \) can be obtained easily. For example, when \( F \) is a uniform distribution, as illustrated in Fig. 2, we have
\[
F(u) = \frac{u - \underline{u}}{\bar{u} - \underline{u}}, \quad u \in [\underline{u}, \bar{u}]
\] (29)
and the inverse function of \( F \) is
\[
F^{-1}(x) = (\bar{u} - u)x + \underline{u}, \quad x \in [0, 1].
\] (30)

Substituting (29) and (30) into (9), we can obtain the service value function as follows:
\[
b(\lambda_n) = \Lambda \cdot \int_{\bar{u}}^{\lambda_n} \frac{x}{\bar{u} - u} dx
\]
\[
= B\lambda_n - C\lambda_n^2
\] (31)
where \( B = \underline{u} \) and \( C = \frac{\underline{u}}{\bar{u} - \underline{u}} \). If we assume \( c(\mu_n) = A\mu_n^2 \), we can easily find that the updated new policy \( \lambda_n^t \) and \( \mu_n^t \) are given as follows:
\[
\lambda_n^t = \begin{cases} 0, & \text{when } \frac{B + G_0(n)}{2C} < 0 \\
\frac{B + G_0(n)}{2C}, & \text{when } 0 \leq \frac{B + G_0(n)}{2C} \leq \Lambda \\
\Lambda, & \text{when } \frac{B + G_0(n)}{2C} > \Lambda 
\end{cases}
\] (32)
\[
\mu_n^t = \begin{cases} \frac{G_0(n)}{2A}, & \text{when } 0 \leq \frac{G_0(n)}{2A} \leq U \\
U, & \text{when } \frac{G_0(n)}{2A} > U 
\end{cases}
\] (33)
where \( G_0(n) = \sum_{i=1}^{M} q_{0i} [g(n_{i+1}) - g(n)] \) and \( G_i(n) = \sum_{j=0}^{M} q_{ij} [g(n_{i,j+1}) - g(n)] \).

For another example, if the service value \( u \) obeys an exponential distribution with mean \( \bar{u} \), we can similarly derive
\[
b(\lambda_n) = \Lambda \cdot \int_{-\bar{u}}^{\lambda_n} x \cdot e^{-x} dx
\]
\[
= \bar{u}\lambda_n \left[ 1 - \ln \left( \frac{\lambda_n}{\Lambda} \right) \right].
\] (34)
Similar policy update formulas may also be derived.

**Case 2:** Now, we consider another case where customers’ service value is identical, i.e., \( u \) is deterministic. In this case, the service value function \( b(\lambda_n) \) is
written as
\[ b(\lambda_n) = \lambda_n \cdot u. \] (35)

If the cost rate function is also of linear form such as \( c(\mu_i, n) = A\mu_i + \lambda_i \) and \( A\mu_i + \lambda_i \) become linear functions as \( G_0(n)\lambda + w\lambda_n \) and \( G_i(n)\mu_i - A\mu_i \), respectively. Thus, the updated arrival rate \( \lambda'_n \) and service rate \( \mu'_i, n \) always have the following form:
\[ \lambda'_n = \begin{cases} \Lambda, & \text{when } u + G_0(n) > 0 \\ 0, & \text{when } u + G_0(n) < 0 \end{cases} \] (36)
\[ \mu'_i, n = \begin{cases} U, & \text{when } G_i(n) - A > 0 \\ 0, & \text{when } G_i(n) - A < 0. \end{cases} \] (37)

So the optimal policy in the linear case is a bang–bang control: the optimal arrival rate \( \lambda^*_n \) and service rate \( \mu^*_i, n \) can be found at the boundary of value domains, i.e., either the maximum or the minimum. This result is consistent with the previous results in the literature [15]–[17]. If the arrival rate or the service rate depend on the work load instead of the system state, i.e., they are denoted as \( \lambda_{n_0} \) and \( \mu_{i, n} \), respectively, the bang–bang control can be further simplified as a threshold policy: when \( n_0 \geq \theta_i \) or \( n_i \geq \theta_i, \lambda^*_n = \Lambda \text{ or } \mu^*_i, n = U; \text{ otherwise, } \text{ they should be zero, } \) where \( \theta_i \) s are thresholds. The similar result can be found in our previous studies [17], [24] and we omit the details in this paper due to limited space.

Summarizing the above-mentioned discussions, we can develop an iterative algorithm to find the optimal policy \((\lambda^*, \mu^*)\) for this rate-setting problem, which is stated by Algorithm 1.

Algorithm 1 is a policy iteration type algorithm for the long-run average performance in the MDP theory, which can also be derived from the Bellman optimality equation. The original optimization problem is decomposed into a series of subproblems (38) and (39). When \( b(\lambda_n) \) and \( c(\mu_i, n) \) are given, such as the form given in Case 1 or Case 2, these subproblems are easy to solve by using analytical methods.

Similar to the classical policy iteration algorithms, we can prove the convergence of Algorithm 1 under proper conditions. The details are omitted for simplicity. Furthermore, Algorithm 1 is also expected to have a fast convergence speed in most of cases. The computation of \( g(n) \) s is very fundamental to implement Algorithm 1 in practice, especially, when we consider the curse of dimensionality in large-scale scenarios [25], [26]. How to extend our research to a large-scale scenario through approximation ways, such as neural networks, deserves further investigation.

### B. Price-Setting Problem

After we obtain the optimal arrival rates \( \lambda^*_n \) and service rates \( \mu^*_i, n \) for Problem 1, we have to solve Problem 2 to find the optimal prices that can induce the same effective arrival rates \( \lambda^*_n \).

#### Algorithm 1: An Iterative Algorithm to Find the Optimal State-Dependent Arrival Rates and Service Rates for Problem 1.

**Initialization**

- Arbitrarily choose an initial policy \((\lambda^0, \mu^0)\), set \( l = 0 \).

**Policy Evaluation**

- Under the current policy \((\lambda^l, \mu^l)\), numerically compute the performance potential \( g^l \) based on (28).

**Policy Improvement**

- Generate a new policy as \((\lambda^{l+1}, \mu^{l+1}) = (\lambda^l, \mu^l) \oplus (\lambda^l, \mu^l) \oplus 1 \oplus \lambda^l, \mu^l \) when \( \lambda^l, \mu^l \in [0, \Lambda], \mu^l, \mu^l \in [0, U], i = 1, \ldots, M, n \in S \):

\[
\begin{align*}
\lambda^l+1 &= \arg\max_{\lambda \in [0, \Lambda]} \left\{ \lambda \sum_{i=1}^M q_i [g\left(n_{i+1} \right) - g\left(n_i \right)] + b(\lambda_n) \right\}, n \in S, \\
\mu^l+1 &= \arg\max_{\mu \in [0, U]} \left\{ \mu \sum_{i=1}^M q_i [g\left(n_{i-1} \right) - g\left(n_i \right)] - g\left(n_i \right) - c(\mu_i, n) \right\}, i = 1, \ldots, M, n \in S.
\end{align*}
\]

To avoid policy oscillation, we choose \( \lambda^l+1 = \lambda^l, \mu^l+1 = \mu^l, n \) if \( \lambda^l, \mu^l \) can also achieve the maximum in (38) (39).

**Stopping Rule**

- If \((\lambda^l+1, \mu^l+1) \neq (\lambda^l, \mu^l)\), set \( l = l + 1 \) and go to step 2; otherwise stop iteration.

According to (12), the optimal price \( p^*_n \) under the optimal policy \((\lambda^*, \mu^*)\) can be computed as follows:

\[
p^*_n = F^{-1} \left( 1 - \frac{\lambda^*_n}{\Lambda} \right) - v \cdot D^*_n, \quad n \in S. \quad (40)
\]

In (40), the distribution function \( F \) is given based on history observations and \( \lambda^*_n \) is given by the optimal policy \((\lambda^*, \mu^*)\). However, the value of the expected delay \( D^*_n \) is unknown. We have to compute \( D^*_n \) before we apply (40) to compute the optimal prices \( p^*_n \). Below, we give a recursive algorithm to compute the expected delay \( D^*_n \) given the system state \( n \) under the optimal policy \((\lambda^*, \mu^*)\).

The expected delay \( D^*_n \) can be viewed as a conditional expectation of sojourn time that an admitted customer will experience if it observes the system state as \( n \) before its entrance. There does not exist any closed-form solution for \( D^*_n \) in the literature. It is difficult to directly compute the value of \( D^*_n \). In this paper, we first compute another quantity that is the expected sojourn time of a customer who gets service from server \( i \) at state \( n+j, i = 1, \ldots, M \). This is because the admission of a customer to the system at state \( n \) results in a state transition from \( n \) to \( n+j \). Then, we develop a recursive method to solve the state-dependent expected sojourn time in semiopen Jackson networks.
networks. Denote $W_{n,k}$ as the expected sojourn time of the $k$th customer at server $i$, given state $n$. We derive the following theorem that provides a numerical way to recursively compute the values of $W_{n,k}$, where $i = 1, \ldots, M$, $k = 1, \ldots, n$.

**Theorem 4**: Under the optimal policy $(\lambda^*, \mu^*)$, the expected sojourn time $W_{n,k}$ is the unique solution to the following set of linear equations:

\[
W_{n,k} = \begin{cases} 
\frac{1}{\lambda_n^*} I_{n_0 > 0} + \sum_{j=1}^{M} \mu_j^* n_j I_{n_j > 0} \\
+ \sum_{m=1}^{M} \frac{\lambda_m^* I_{n_0 > 0} + \sum_{j=1}^{M} \mu_j^* n_j I_{n_j > 0}}{\mu_m^* n_m \cdot n_{m+1}} W_{n,k+m} \\
+ \sum_{m=1, m \neq j=0, j \neq m}^{M} \frac{\mu_m^* n_m I_{n_m > 0} q_{mn}}{\lambda_n^* I_{n_0 > 0} + \sum_{j=1}^{M} \mu_j^* n_j I_{n_j > 0}} W_{n,k+m-j} \\
if k = 1 \end{cases} \tag{41a}
\]

\[
W_{n,k} = \begin{cases} 
\frac{1}{\lambda_n^*} I_{n_0 > 0} + \sum_{j=1}^{M} \mu_j^* n_j I_{n_j > 0} \\
+ \sum_{m=1}^{M} \frac{\lambda_m^* I_{n_0 > 0} + \sum_{j=1}^{M} \mu_j^* n_j I_{n_j > 0}}{\mu_m^* n_m \cdot n_{m+1}} W_{n,k+m} \\
+ \sum_{m=1, m \neq j=0, j \neq m}^{M} \frac{\mu_m^* n_m I_{n_m > 0} q_{mn}}{\lambda_n^* I_{n_0 > 0} + \sum_{j=1}^{M} \mu_j^* n_j I_{n_j > 0}} W_{n,k+m-j} \\
if 2 \leq k \leq n_i. \tag{41b}
\end{cases}
\]

Note that we have $W_{n,k}^{m-1, j} = W_{n,k}^{m, j}$ if $j = 0$ in (41a) and (41b), and $W_{n,k-1}^{m, j} = W_{n,k}^{m, j}$ if $m = 0$ in (41b).

**Proof**: The above recursive equations are derived based on the analysis of one-step transition of the Markov process. By moving every term associated with $W_{n,k}$ on one side and the term $-\frac{1}{\lambda_n^*} I_{n_0 > 0} + \sum_{j=1}^{M} \mu_j^* n_j I_{n_j > 0}$ to the other side, we can rewrite the above linear equations in a matrix form $AW = C$. $A$ is the coefficient matrix when everything is moved to the right side of (41a) and (41b), $W$ is a column vector composed of $W_{n,k}$, $s$, $i = 1, \ldots, M$, $k = 1, \ldots, n_i$, $n \in S$. $C$ is a column vector whose element is $-\frac{1}{\lambda_n^*} I_{n_0 > 0} + \sum_{j=1}^{M} \mu_j^* n_j I_{n_j > 0}$, if the element in the same position in $W$ is $W_{n,k}$. When $k = 1$, for the corresponding row vector in $A$, the negative of diagonal element is $\sum_{j=1}^{M} \mu_j^* n_j I_{n_j > 0}$ bigger than the sum of all the remaining elements in the same row, the row is strictly diagonally dominant. When $2 \leq k \leq n_i$, for the corresponding row vector in $A$, the negative of diagonal element is equal to the sum of all the remaining elements in the same row, the row is weakly diagonally dominant. For any weakly diagonally dominant row, we can find at least one walk in the directed graph of $A$ from the weakly diagonally dominant row to a strictly diagonally dominant row. So $A$ is a weakly chained diagonally dominant matrix, such matrix is nonsingular, according to the theorem in [27]. Thus, the unique solution of this set of linear equations is guaranteed.

The above theorem gives a recursive and numerical way to compute the expected sojourn time of every customer in the system at any given state. For determining the expected delay of a customer arriving at state $n$, we only need the expected sojourn time of the $(n+1)$th customer at server $i$, given state $n_i$, for all $i$. That is, the value of $D_{n, \lambda}^*$ is computed as follows:

\[
D_{n, \lambda}^* = \sum_{i=1}^{M} q_{n_i} W_{n_i, n_i+1}^{n_i, j}. \tag{42}
\]

Therefore, the optimal price $p_{n_i}^*$ in this price-setting problem can be obtained by using (40), (42), and Theorem 4. Consequently, the original pricing and service rate control problem (Problem 0) has been solved, according to Theorem 1. The optimal price $p_{n_i}^*$ and service rate $\mu_{n_i}^*$, $i = 1, \ldots, M$, $n \in S$, are obtained by (40) and Algorithm 1, respectively.

In this paper, the pricing control problem is dynamic, and we obtain the optimal prices and service rates at every system state. That is, we can observe the full information of the system and adjust the price and service rates dynamically when the system state is changed. If we can only observe partial information to adjust the policy, this problem may become more complicated. For example, we may only observe the total number of customers in the whole network, instead of the specific distribution of customers. We adjust the price only when the total number of customers is changed. Such partial information based pricing control problem deserves further investigation, and our previous study on admission control problem may provide some insights [24]. Furthermore, if the price and service rates are not varying with state, we call it static pricing control problem. Such problem may become even more complicated and gradient-based approaches may be applicable. All these problems are future research topics with significance.

**IV. NUMERICAL EXPERIMENTS**

In this section, we explore the characteristics of the optimal policy derived by our approach through some numerical examples. We also discuss the impact of the variation of delay cost parameters on the system performance.

Consider a semiopen Jackson network with $M = 2$ servers. The routing probabilities are $q_{01} = 0.3$, $q_{02} = 0.2$, $q_{10} = 0.8$, $q_{12} = 0.2$, $q_{20} = 0.4$, and $q_{21} = 0.6$. Obviously, we have $\lambda_0 = 0$ if $\sum_{i=1}^{M} n_i = N$ and $\mu_{i,n} = 0$ if $n_i = 0$, $i = 1, \ldots, M$. We assume that the service values are uniformly distributed in $[\mu_{i}, \mu_{i,n}]$. As mentioned before, the service rate function $b(\lambda_n) = B_{n} - C(\lambda_n)$ and the marginal value function $b'(\lambda_n) = B - 2C_{\lambda_n}$, where $B = \pi$ and $C = \frac{b_{\lambda_n}}{2}$. The operating cost rate function is $c(\mu_{i,n}) = \frac{1}{2} \mu_{i,n}^2$. The arrival rate $\lambda_n$ is chosen from $[0, \Lambda]$ and the service rate $\mu_{i,n}$ is chosen from $[0, U]$, $i = 1, \ldots, M$ and $n \in S$.

First, we investigate the optimality structure of the state-dependent arrival rates, service rates, and expected delays.

As we can see from Fig. 3(a), the optimal arrival rate $\lambda_n^*$ is
nonincreasing in state \( n \), that is, if \( n_1 \geq n_2 \), we have \( \lambda^*_n \leq \lambda^*_{n_2} \). The optimal arrival rate \( \lambda^*_n \) decreases to zero before the total number of customers reaches 20. This indicates that the system stability is achieved via pricing control under the rational customer behavior mechanism, which justifies the statement of the finite capacity assumption in Remark 1. In Fig. 3(b) and (c), we can see that the optimal service rate of each server is increasing with the number of customers at that server if we keep the number of customers at the other server fixed. The expected delay \( D^*_{n_1} \) is increasing in state \( n \), as illustrated in Fig. 3(d).

Next, we explore the structure of optimal state-dependent price \( p^*_n \) for positive optimal arrival rate \( \lambda^*_n \). With (40), we know that the price at state with \( \lambda^*_n = 0 \) can be any value equal to or larger than \( \pi - v \cdot D^*_n \), so we only focus on the structure of optimal price \( p^*_n \) with positive arrival rate \( \lambda^*_n \) in the following examples. When the delay cost \( v \) is relatively small as compared to the service value, we have \( p^*_n \approx b'(\lambda^*_n) \). As the value rate function \( b(\lambda^*_n) \) is concave and the optimal arrival rate is decreasing in \( n \) when \( \lambda^*_n \neq 0 \), the optimal price \( p^*_n \) is increasing in state \( n \), as illustrated in Fig. 4(a). In a light-traffic scenario where the maximal arrival rate \( \Lambda \) is small, the optimal state-dependent arrival rate \( \lambda^*_n \) reaches the maximum at states where the total number of customers \( |n| \) is small, and is decreasing in state \( n \) when \( |n| \) is large. As a result, the optimal price \( p^*_n \) is decreasing in state \( n \) when \( |n| \) is small and increasing in state \( n \) when \( |n| \) is relatively large. Fig. 4(b) is an example for this nonmonotone structure of \( p^*_n \).

Then, we examine the effect of delay cost \( v \) on the optimal policy. Due to space limitations, we only plot the optimal policy for the case where server 2 is idle. The following results also hold for any positive number of customers in server 2. From Fig. 5(a), we can see that when server 2 is idle, for the same state \( n \), the optimal arrival rate \( \lambda^*_n \) is nonincreasing in delay cost \( v \). The maximal number of customers at server 1 with positive arrival rate is decreasing in delay cost \( v \). This suggests that higher delay cost decreases the maximal number of customers in the system. To counteract the effect of the increased delay cost on the system performance, the service provider has to increase the service rates of servers to reduce the expected delay. So, we see that for the same state \( n \), the optimal service rate \( \mu^*_2,n \) is nondecreasing in delay cost \( v \) in Fig. 5(b), and that the expected delay \( D^*_n \) is decreasing in delay cost \( v \) in Fig. 5(c). The optimal service rate \( \mu^*_2,n \) of server 2 under varying delay cost shares...
the same monotonicity property with that of server 1. We omit its plot for space limitation. The variation of optimal price as a result of the change in delay cost is relatively small, but there is no unified conclusion about whether an increased delay cost raises prices, as shown by Fig. 5(d).

In addition, we study the long-run average profits of customers and service provider if the policy is optimal and the value of \( v \) is varying. The profit of customers is also called the expected net utility defined in (1). Thus, the long-run average profit of customers admitted to the system is given by

\[
r_c = \lim_{T \to \infty} \frac{1}{T} \int_0^T \lambda_n R_n dt = \sum_{n \in S} \pi(n) \lambda(n) [E\{u|u > \bar{u}_n\} - p_n - vD_n].
\]

The profit of the service provider is defined as the revenue gained from charging customers minus the operating cost. Thus, at state \( n \), the profit rate of the service provider is written as

\[
S_n = \lambda_n p_n - \sum_{i=1}^M c(\mu_i n).
\]

The long-run average profit of the service provider is written as

\[
r_s = \lim_{T \to \infty} \frac{1}{T} \int_0^T S_n dt = \sum_{n \in S} \pi(n) S_n.
\]

Table I gives the optimal values of \( r^*_c \), \( r^*_s \), and \( \eta^* \) under different delay cost \( v \). We set capacity \( N \) loosely as 30. The optimal arrival rates under this set of delay cost parameters all diminish to zero before the total number of customers in the system reaches 30. From Table I, we see that the values of \( r^*_c \), \( r^*_s \), and \( \eta^* \) are all decreasing in \( v \). Moreover, we can find that the sum of \( r^*_c \) and \( r^*_s \) equals the long-run average social welfare \( \eta^* \). Based on the above-mentioned analysis, we can conclude that if customers have a relatively lower \( v \), it not only enlarges their own expected profits, but also contributes to the growth of the profit of service provider and the social welfare. It explains the English saying “patience is a virtue” from a scientific viewpoint of service systems. Useful insights may be obtained for the service operation management in practice: patient customers are beneficial to all participants in a service system and the service regulator should lower the weight of delay cost if possible. There are many ways widely adopted in practice to improve the customer patience and lower the value of \( v \). For example, service providers can furnish the waiting area comfortably or providing interesting games to make waiting tolerable, pleasant, or even productive. Some psychological techniques are also beneficial to improve the customer patience.

### V. CONCLUSION

In this paper, we study the dynamic pricing and service rate control problem in a semiopen Jackson network, where the customers are delay sensitive and rational. To handle the difficulty caused by the pricing control, we decompose the original

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problem into the rate-setting problem and the price-setting problem. Based on the performance difference formulas, we first develop the iterative algorithm to optimize the state-dependent arrival rates and service rates. When the value rate function is concave in the arrival rate and the cost rate function is convex in the service rate, the optimal state-dependent arrival and service rates are solved by a series of convex optimization subproblems. When the rate functions have linear structure, the optimal rates are of a bang–bang control form. Then, we develop a recursive approach to solve the conditional expected delays in the price-setting problem and obtain the optimal prices consequently. Finally, we conduct numerical examples to demonstrate the structure property of optimal policies. Interesting sights are also derived, which may guide the service management in real life: patience of customers is beneficial to the performance of customers, service provider, and even the social welfare; we should lower the weight of delay cost by adopting strategies for managing customer waiting.

In this paper, we assume that all the customers have the same admission prices and value functions. However, in many systems, there exist multiple classes of customers. Different classes of customers may have different prices and rate functions. Thus, their admission decisions will be mutually affected and there exist competitions among customers. It is an interesting future topic to study the pricing control problem from a game theoretic viewpoint. Other future topics may include the dynamic pricing control based on partial information and the static pricing control.

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REFERENCES