Outline

• Buzen’s algorithm
• Mean value analysis for Jackson networks
• Cyclic network
• Extension of Jackson networks
• BCMP network
Marginal Distributions based on Buzen’s algorithm

• With Buzen’s algorithm, we can easily calculate marginal distributions
  – For single server with load-independent rate
    • Marginal distribution
      \[
p(n_i \geq k) = \frac{\rho_i^k g(M, N-k)}{G(N)}
      \]
    • Marginal probability
      \[
p(n_i = k) = \rho_i^k g(M, N-k) - \rho_i g(M, N-k-1)
      \]
      \[
p(n_M = k) = \frac{\rho_M^k g(M -1, N-k)}{G(N)}
      \]
Performance metrics based on Buzen’s algorithm

• Other performance metrics
  – For single server with load-independent rate
  • Average number of customers:
    \[ E[n_i] = \sum_{k=1}^{N} \rho_i^k \frac{g(M, N-k)}{G(N)} \]
  • Throughput of server i:
    \[ \text{TH}_i = \mu_i p(n_i \geq 1) = \mu_i \rho_i \frac{g(M, N-1)}{G(N)} \]
  • True value of visit ratio \( v_i \), satisfied
    \[ v_i = \sum_{j=1}^{M} v_j r_{ji} \]
    \[ v_i = \text{TH}_i = \mu_i \rho_i \frac{g(M, N-1)}{G(N)} \]
Calculation based on Buzen’s algorithm

- Calculation procedure
  - Calculate Buzen’s table
  - Based on the table, calculate the distribution or performance metrics

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>m−1</th>
<th>m</th>
<th>...</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>...</td>
<td>1</td>
<td>1</td>
<td>...</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$\rho_1$</td>
<td>$\rho_2 + \rho_1$</td>
<td></td>
<td>...</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$\rho_1^2$</td>
<td>$\rho_1^2 + \rho_2(\rho_2 + \rho_1)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$\rho_1^3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>n−1</td>
<td>$\rho_1^{n−1}$</td>
<td></td>
<td></td>
<td></td>
<td>$g(m,n−1)$</td>
<td>(\downarrow \rho_m)</td>
<td>$g(m,n)$</td>
</tr>
<tr>
<td>n</td>
<td>$\rho_1^n$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td>$\rho_1^N$</td>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Mean value analysis

• Two methods to analyze closed Jackson network
  – Buzen’s algorithm to compute $G(N)$ and distributions
  – Mean value analysis to recursively compute the average performance metrics, not distributions

• Mean value analysis, proposed by
Arrival theorem

• Arrival theorem
  – A general case of PASTA theorem
  – Also called random observer property (ROP) or job observer property
  – “upon arrival at a station, a job observes the system as if in steady state at an arbitrary instant for the system without that job”
Arrival theorem

• Applicability condition
  – PASTA, Poisson arrival
  – always hold in open product form networks with unbounded queues at each node (not Poisson arrival)
  – other general networks (not Poisson arrival)
  – for open Jackson network
    • \( q(n) = p(n) \)
  – For closed Jackson network
    • \( q_i(N,n_{-i}) = p(N-1,n_{-i}) \), the statistics seen by an arrival customer is equal to that of the steady network with one customer less.
Mean value analysis

• Based on two basic principles
  – Arrival theorem (PASTA or ROP for Jackson networks)
  – Little’s law

• For closed Jackson network
  – $q_n(N) = p_n(N-1)$, queue length seen by arrival equals that of the network with one less customer
  – Little’s law is applicable throughout the network
Mean value analysis

• With ROP
  – $W_i(N) = \frac{[1+L_i(N-1)]}{\mu_i}$ (also valid for M/M/1 or M/M/c)
    • $W_i(N)$: mean response time at node $i$ for a network with $N$ customers
    • $L_i(N-1)$: average number of customers at node $i$ for a network with $N-1$ customers

• With Little’s law
  – $L_i(N) = \lambda_i(N)W_i(N)$
    • $\lambda_i(N)$: throughput (arrival rate) of node $i$ in an $N$-customer network, which is unknown
Calculation of $\lambda_i(n)$

- Calculate visit ratio $v_i$ by traffic equations
  
  $v_i = \sum_{j=1}^{M} v_j r_{ji}$, for all $i = 1,\ldots,M$

  - $v_i$ is the relative throughput of node i

- Calculate $\lambda(n)$: 
  
  $\lambda(n) = \frac{n}{\sum_{i=1}^{M} v_i W_i(n)}$

- Throughput of node i: 
  
  $\lambda_i(n) = \lambda(n) v_i$
Algorithm of mean value analysis

• Solve traffic equations to obtain $v_i$, $i=1,2,...,M$
• Initialize $L_i(0)=0$, $i=1,2,...,M$
• For $n=1:N$, calculate
  - $W_i(n) = (1+L_i(n-1))/\mu_i$
  - $\lambda(n)=n/[v_1W_1(n)+...+v_MW_M(n)]$
  - $\lambda_i(n)=\lambda(n)v_i$, $i=1,2,...,M$
  - $L_i(n) = \lambda_i(n)W_i(n)$, $i=1,2,...,M$
Discussion of mean value analysis

• Only calculate the average metrics
  – Average queue length, mean waiting time, mean response time
  – Cannot calculate the steady state distribution

• Recursive algorithm
  – Complexity is linear to the system size

• For multiclass networks
  – Also applicable, but complexity grows exponentially with the number of classes
Example

• Similar to Example 4.5 in page 203 of textbook
  – Closed Jackson network with M=3, N=2
  – Exception: all the node are single server
  – Service rate: $\mu_1=2$, $\mu_2=1$, $\mu_3=3$
  – Routing prob.: $r_{12}=3/4$, $r_{13}=1/4$, $r_{21}=2/3$, $r_{23}=1/3$, $r_{31}=1$
Example (cont.)

• Visit rate equation set: 
  \[ v_1 = \frac{2}{3} v_2 + v_3 \]
  Let \( v_1 = 1 \), solve the equation
  \[ v_2 = \frac{3}{4} v_1 \]
  \( v_2 = 3/4, \ v_3 = 1/2 \)

• State space: \((M+N-1)\) choose \(N\), it is 6
  \((0,0,2), (0,1,1), (0,2,0), (1,0,1), (1,1,0), (2,0,0)\)

• Buzen’s algorithm: \(G(N) = 2.3086\)
  \( \pi = (0.0214, 0.0962, 0.4332, 0.0642, 0.2888, 0.0962) \)
Example (cont.)

• MVA method
  – For n=1:
    • $W_1(1) = \frac{1+L_1(0)}{\mu_1} = 1/2$; $W_2(1) = 1$; $W_3(1) = 1/3$
    • $\lambda(1) = \frac{1}{1*1/2+3/4*1+1/2*1/3} = 12/17$
    • $\lambda_1(1) = \lambda(1)v_1 = 12/17$; $\lambda_2(1) = 9/17$; $\lambda_3(1) = 6/17$
    • $L_1(1) = \lambda_1(1)W_1(1) = 6/17$; $L_2(1) = 9/17$; $L_3(1) = 2/17$
  – For n=2:
    • $W_1(2) = \frac{1+L_1(1)}{\mu_1} = 23/34$; $W_2(2) = 26/17$; $W_3(2) = 19/51$
    • $\lambda(2) = \frac{1}{1*23/34+3/4*26/17+1/2*19/51} = 102/205$
    • $\lambda_1(2) = \lambda(2)v_1 = 102/205$; $\lambda_2(2) = 153/410$; $\lambda_3(2) = 51/205$
    • $L_1(2) = \lambda_1(2)W_1(2) = 69/205$; $L_2(1) = 117/205$; $L_3(1) = 19/205$
Thinking of closed Jackson network

• Relation of parameters
  – Arrival rate (throughput) $\lambda$ v.s. service rates $\mu$
    • $\mu$ increases, $\lambda$ increases
  – Arrival rate $\lambda$ v.s. number of customers $N$
    • $N$ increases, $\lambda$ increases with a upper bound

• MVA for marginal distribution
  – Recursive formula, similar to M/M/1
    $p_i(n,N)=\frac{\lambda_i(N)}{\mu_i} * p_i(n-1,N-1)$
Cyclic network

• A special case of closed Jackson network
  – $r_{ij}=1$, if $j=i+1$ and $0<i<M$; $r_{M1}=1$; otherwise $r_{ij}=0$
Cyclic network

- Product-form solution of steady state distribution
  - Traffic equation is special, $v_{i+1} = v_i$, so set all $v_i = 1$
  - $\rho_i = v_i / \mu_i = 1 / \mu_i$

$$p_n = \frac{1}{G(N)} \rho_1^{n_1} \rho_2^{n_2} \cdots \rho_M^{n_M} = \frac{1}{G(N) \mu_1^{n_1} \mu_2^{n_2} \cdots \mu_M^{n_M}}$$

where

$$G(N) = \sum_{n_1 + \cdots + n_M = N} \frac{1}{\mu_1^{n_1} \mu_2^{n_2} \cdots \mu_M^{n_M}}$$
Extension of Jackson networks

• Load-dependent arrival rate and service rate
  – Similar results, product form solution

• Consider travel time between nodes
  – Model the travel time as extra nodes with ample servers, still keep the form of Jackson networks

• Multiclass Jackson network
  – Each class has its own routing structure, arrival rates and service rates
  – Applicable for computer, communication systems.
  – BCMP network, still have product-form solution
Non-Jackson network

• Many variants from Jackson networks

• State-dependent routing probability
  – Customer has flexibility to decide its next stop
    • E.g., choose the node with less congestion
  – Even exponential interarrival and service time, no product-form solution
  – Use Markov model to do analysis, but suffer from “curse of dimensionality”
    • Product-form solution avoids this curse of computation
    • Storage is a curse if needs store every state distribution
      – Avoid to store every distribution, use iterative calculation, e.g., for all s: \( L = L + n \cdot p(s) \), only one iterative variable \( L \)
BCMP network

- BCMP network definition
  - M servers, K classes of customers
  - 4 kinds of service disciplines
    - FCFS, PS, IS (infinite servers, or ample servers), LCFS with preemptive-resume
  - Class transition
    - class k customer from server i transits to server j as class r, with probability $q_{ij,kr}$
  - Service time distribution
    - FCFS: IID exponential for all classes;
    - PS, IS, LCFS: any COX distribution (including exponential)
BCMP network

• Steady state distribution of BMCP network has a product-form solution
  – Handle each server independently and multiply them together
  – Calculate the normalization constant,
    • Does exit similar algorithm to Buzen’s?
  – Scalability, avoid the curse of dimensionality
    • Applicable to large-scale problem